

**PHY 711 Classical Mechanics and
Mathematical Methods**
**10-10:50 AM MWF online or (occasionally) in
Olin 103**
Plan for Lecture 25 – Chap. 8 (F & W)

Motions of elastic membranes

- 1. Review of standing waves on a string**
- 2. Standing waves on a two dimensional membrane.**
- 3. Boundary value problems**

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In this lecture, we will resume our consideration of elastic media, extending the one dimensional analysis of a string to a two dimensional membrane.

10	Wed, 9/16/2020	Chap. 3 & 6	Lagrangian & constraints	#8	9/21/2020
11	Fri, 9/18/2020	Chap. 3 & 6	Constants of the motion		
12	Mon, 9/21/2020	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/23/2020
13	Wed, 9/23/2020	Chap. 3 & 6	Liouville theorem	#10	9/25/2020
14	Fri, 9/25/2020	Chap. 3 & 6	Canonical transformations		
15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020



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The homework assignment relate to rigid body motion.

Thursday's Physics Online Colloquium

<https://www.physics.wfu.edu/wfu-phy-news/seminars-2020-fall/>



Department of Physics

Oct. 22, 2020
at 4 PM



Ali Daraei

Graduate Student
Mentor, Dr. Martin Guthold
Physics Department
Wake Forest University, Winston-Salem, NC

"Intrinsically Unfolded Alpha-C Connector of Fibrinogen is a Major Contributor to the Mechanical Strength of Fibrin Fibers"

Fibrinogen is the key mechanical protein in blood coagulation since it is the building block of fibrin fibers, and these 100 nm thick fibers provide mechanical and structural stability to blood clots as they stem the flow of blood. In hemostasis, blood clots stop blood flow in the event of injury to blood vessels, and they are involved in the initiation of wound healing. In this case, they are beneficial – in fact, lifesaving – for the individual. On the other hand, in thrombosis, the aberrant formation of blood clots inside blood vessels causes serious diseases. For example, blood clots are the underlying pathology of myocardial infarction, ischemic strokes, deep vein thrombosis, and pulmonary embolism. In both scenarios, hemostasis and thrombosis, clots mechanically stop the flow of blood. The major structural

Colloquium

FALL 2020

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Thursday's lecture is by a senior biophysics graduate student. Please consult the webpage for details including relevant references on the topic.

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Review of the wave equation in one special dimension.

Initial value problem : $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int^x \psi(x') dx'$$

For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

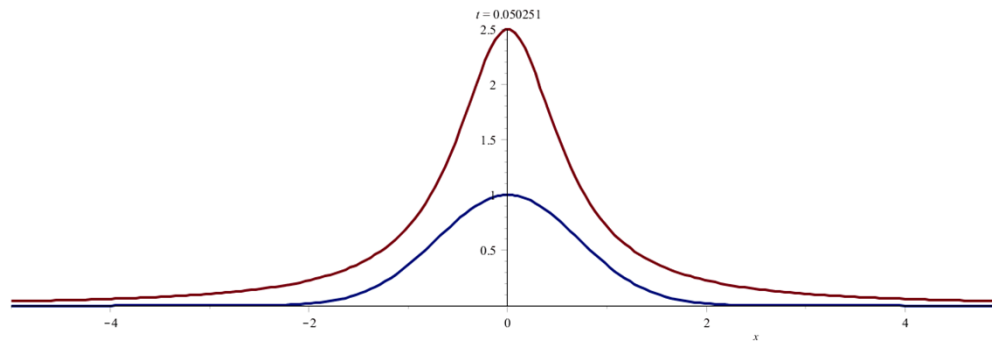
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Review continued.

Example with $\psi(x) = 0$ and $\phi(x) = \frac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Two examples of traveling waves.

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

with $\mu(0, t) = \mu(L, t) = 0$.

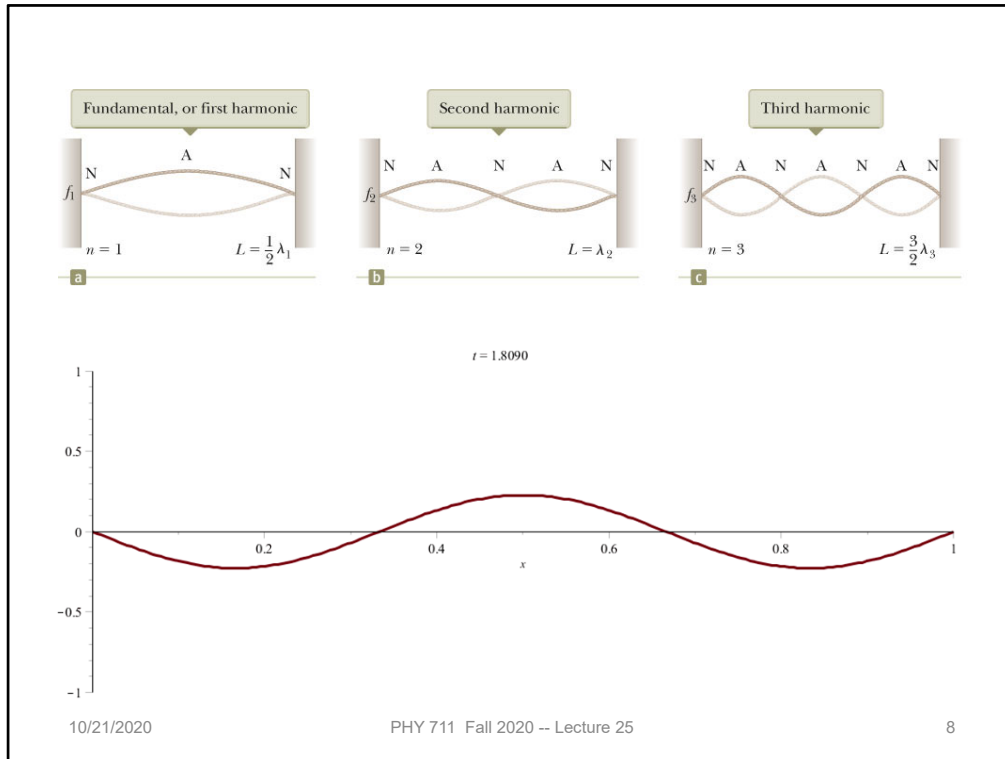
Assume: $\mu(x, t) = \Re(e^{-i\omega t} \rho(x))$

$$\text{where } \frac{d^2 \rho(x)}{dx^2} + k^2 \rho(x) = 0 \quad k = \frac{\omega}{c}$$

$$\rho_v(x) = A \sin\left(\frac{v\pi x}{L}\right)$$

$$k_v = \frac{v\pi}{L} \quad \omega_v = ck_v$$

Standing wave solutions for constrained string.



Some more details of standing waves.

Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

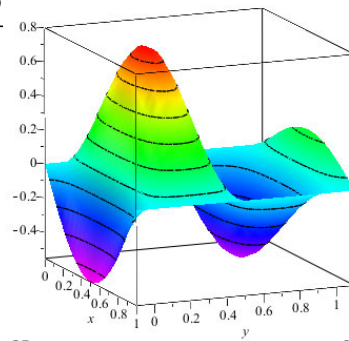
$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re(e^{-i\omega t} \rho(x, y))$$

$$(\nabla^2 + k^2) \rho(x, y) = 0 \quad \text{where } k = \frac{\omega}{c}$$

$$\rho(x, y)$$



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Now consider, the same idea, generalized to two spatial dimensions. Here we will focus on standing wave solutions.

Lagrangian density: $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dx dy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial t)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial x)} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial y)} = 0$$

It is possible to formulate the treatment using a continuous Lagrangian.

Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial x)} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t} \rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right) \rho(x, y) = 0 \quad \text{where } k = \frac{\omega}{c}$$

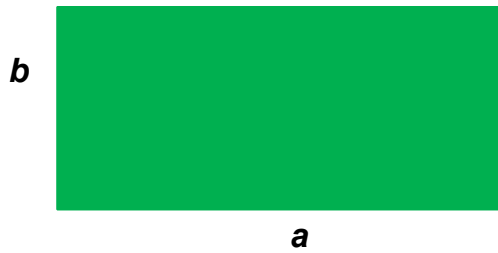
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Some details.

Consider a rectangular boundary:



Clamped boundary conditions :

$$\rho(0, y) = \rho(a, y) = \rho(x, 0) = \rho(x, b) = 0$$

$$(\nabla^2 + k^2)\rho(x, y) = 0$$

$$\Rightarrow \rho_{mn}(x, y) = A \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$\text{where } k = \frac{\omega}{c}$$

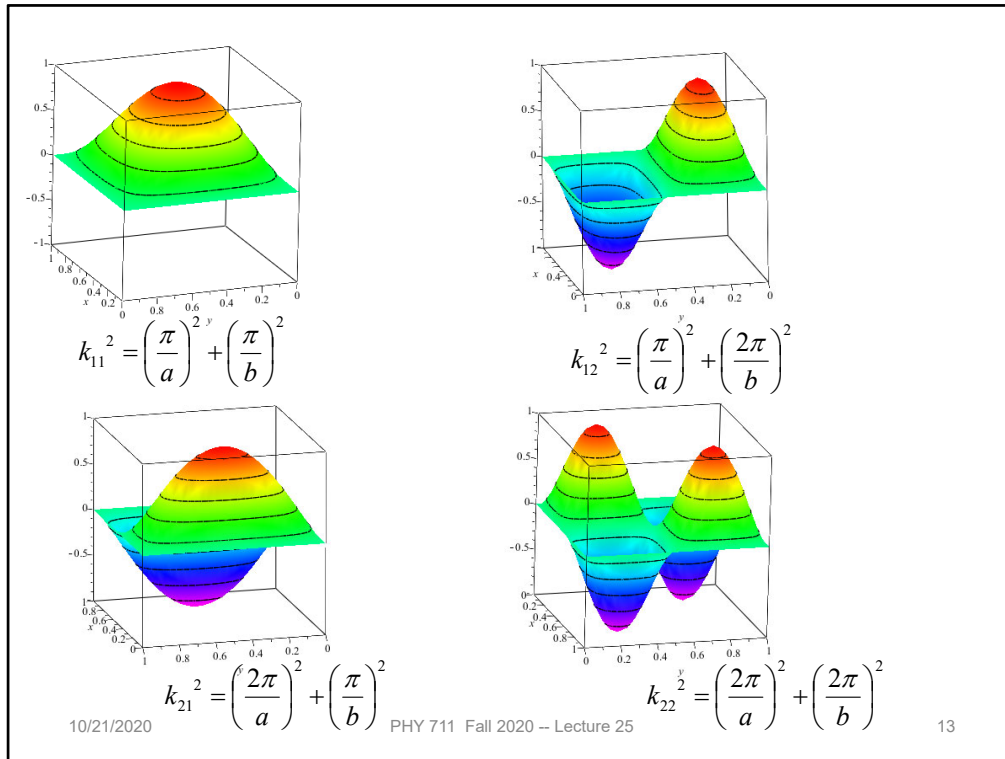
$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

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An example of the rectangular membrane clamped on all edges.



Some two dimensional standing waves.

More general boundary conditions:

$\tau \nabla u|_b = \kappa u|_b$ represents bounded side constrained with spring

$\tau \nabla u|_b = 0$ represents "free" side

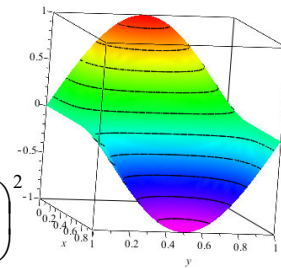
Mixed boundary conditions :

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

$$k_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



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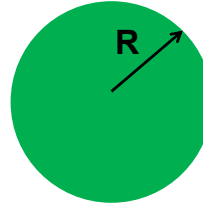
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Other possible boundary conditions.

Consider a circular boundary:

Clamped boundary conditions for $\rho(r, \varphi)$:

$$\rho(R, \varphi) = 0$$



$$(\nabla^2 + k^2)\rho(r, \varphi) = 0 \quad \text{where } k = \frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

$$\text{Assume: } \rho(r, \varphi) = f(r)\Phi(\varphi)$$

$$\text{Let: } \Phi(\varphi) = e^{im\varphi}$$

$$\begin{aligned} \text{Note: } \Phi(\varphi) &= \Phi(\varphi + 2\pi) \\ &\Rightarrow m = \text{integer} \end{aligned}$$

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Another example of a membrane, this time clamped at the boundary of a circle. (Such as in a drum for example.) It is convenient to polar coordinates.

Consider circular boundary -- continued

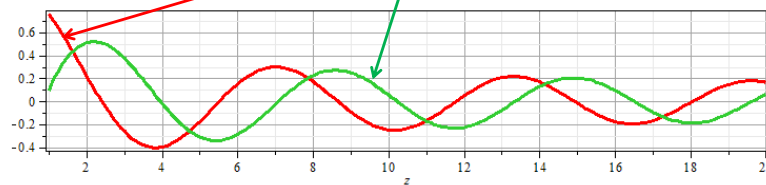
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

\Rightarrow Bessel equation of integer order with transcendental solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



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The radial equation has this special form which is conveniently expressed in terms of Bessel functions.

Some properties of Bessel functions

Ascending series: $J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!} \left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$

$$J_{m-1}(z) - J_{m+1}(z) = 2 \frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z) \xrightarrow{z \gg 1} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$

Zeros of Bessel functions $J_m(z_{mn}) = 0$

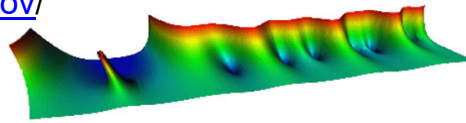
$m = 0$: $z_{0n} = 2.406, 5.520, 8.654, \dots$

$m = 1$: $z_{1n} = 3.832, 7.016, 10.173, \dots$

$m = 2$: $z_{2n} = 5.136, 8.417, 11.620, \dots$

Some properties of Bessel functions of integer order.

<http://dlmf.nist.gov/>



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 [DLMF Update: Version 1.0.9](#)
2014-04-25 [DLMF Update: Version 1.0.8, errata & improved MathML](#)
2014-03-21 [DLMF Update: Version 1.0.7; New Features Improve Math & 3D Graphics](#)
2013-08-16 [Bille C. Carlson, DLMF Author, dies at age 89](#)
[More news](#)

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2 Asymptotic Approximations	23 Weierstrass Elliptic and Modular Functions
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A resource for finding properties of special functions including Bessel functions.

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\nu + k + 1)}.$$

$$Y_n(z) = -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \ln\left(\frac{1}{2}z\right) J_n(z) \\ - \frac{\left(\frac{1}{2}z\right)^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!},$$

Some details.

Some properties of Bessel functions -- continued

Note : It is possible to prove the following

identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right) J_m\left(\frac{z_{mn'}}{R}r\right) r dr = \frac{R^2}{2} (J_{m+1}(z_{mn}))^2 \delta_{nn'}$$

Returning to differential equation for radial function :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

$$\Rightarrow f_{mn}(r) = A J_m\left(\frac{z_{mn}}{R}r\right); \quad k_{mn} = \frac{z_{mn}}{R}$$

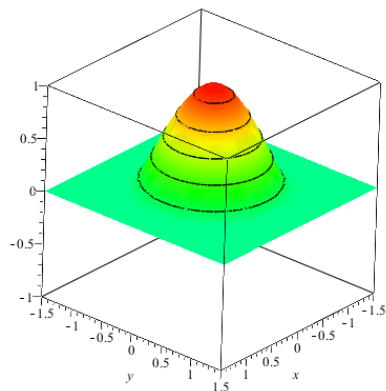
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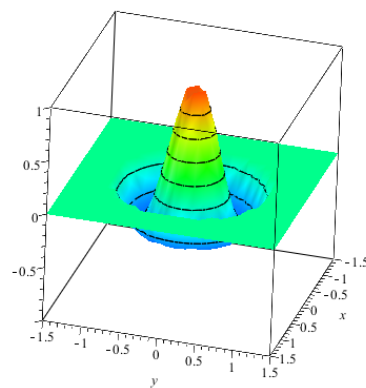
Patient mathematicians worked out lots of useful relationships. We are particularly interested in aligning the zeros of the Bessel functions with the boundaries of our membrane.

$$\rho_{01}(r, \varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right)$$



$$k_{01} = \frac{2.406}{R}$$

$$\rho_{02}(r, \varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$



$$k_{02} = \frac{5.520}{R}$$

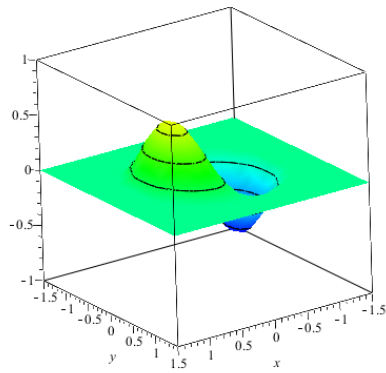
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Some examples.

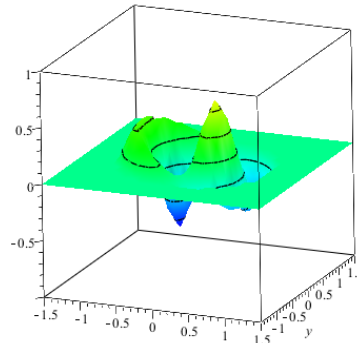
$$\begin{aligned}\rho_{11}(r, \varphi) &= f_{11}(r) \cos(\varphi) \\ &= AJ_1\left(\frac{z_{11}}{R} r\right) \cos(\varphi)\end{aligned}$$



$$k_{11} = \frac{3.832}{R}$$

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$$\begin{aligned}\rho_{12}(r, \varphi) &= f_{12}(r) \cos(\varphi) \\ &= AJ_1\left(\frac{z_{12}}{R} r\right) \cos(\varphi)\end{aligned}$$



$$k_{12} = \frac{7.016}{R}$$

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More examples.

Ernst Chladni



Ernst Chladni

Born 30 November 1756
[Wittenberg, Electorate of Saxony](#)
in the [Holy Roman Empire](#)

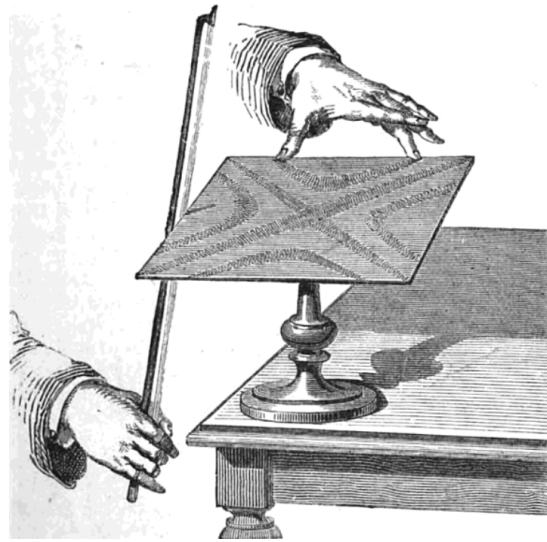
Died 3 April 1827 (aged 70)
[Breslau, Province of Silesia](#) in the
[Kingdom of Prussia](#), a part of the
[German Confederation](#)

Nationality [German](#)

Known for [Study of acoustics](#)
[Chladni plates and figures](#)
[Estimating the speed of sound](#)
[Chladni's law](#)
[Theory of meteorites' origins](#)

Scientific career

Fields [Physics](#)



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A very nice demonstration of these standing waves was invented by Chladni

Demonstration with motor in the middle – (PASCO)



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A picture of the demo we have in Olin.

<http://www.physics.wfu.edu/resources/education-resources/demo-videos/waves/>



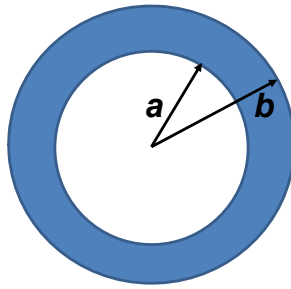
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Movie thanks to Eric Chapman.

More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r, \varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$
 $\Rightarrow m = \text{integer}$

A non-trivial example with two boundaries.

Consider circular boundary -- continued

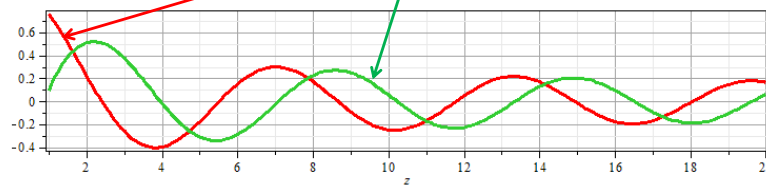
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

\Rightarrow Bessel equation of integer order with transcendental solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



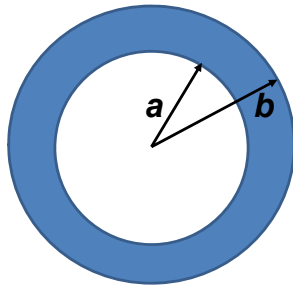
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In this case, both Bessel and Neumann functions are needed.

Normal modes of an annular membrane -- continued



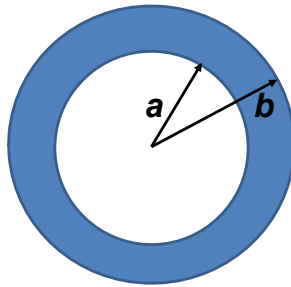
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

General form of radial function: $f(r) = AJ_m(kr) + BN_m(kr)$

We need to find the linear coefficients A and B and the wavevector k.

Normal modes of an annular membrane -- continued



Boundary conditions:

$$f(a) = 0 \quad f(b) = 0$$

$$AJ_m(ka) + BN_m(ka) = 0$$

$$AJ_m(kb) + BN_m(kb) = 0$$

\Rightarrow 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)} \quad (\text{transcendental equation for } k)$$

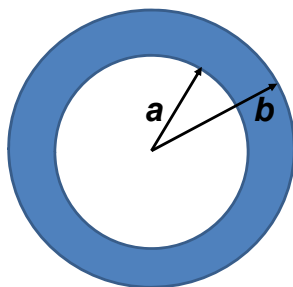
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A method of solving this problem.

Normal modes of an annular membrane -- continued



Boundary conditions:

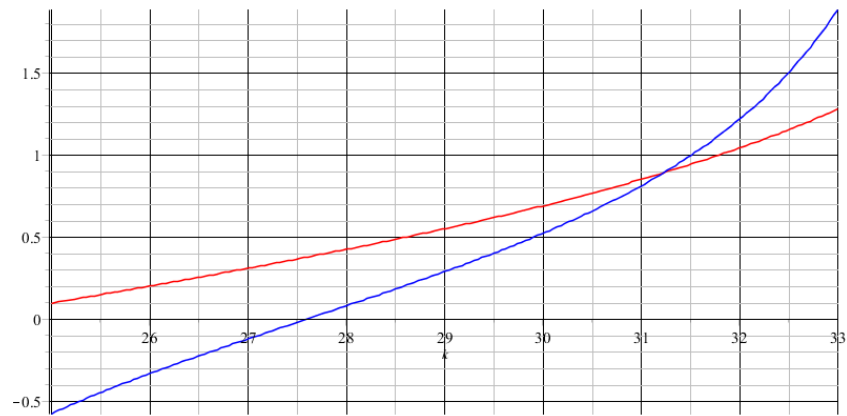
$$f(a) = 0 \qquad f(b) = 0$$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)} \quad \text{-- in terms of solution } k_{mn} :$$

$$f(r) = A \left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right)$$

Analysis for $m=0$ and $a=0.1$, $b=0.2$:

```
=
> plot( { -BesselJ(0, 0.1*k) / BesselY(0, 0.1*k), -BesselJ(0, 0.2*k) / BesselY(0, 0.2*k) }, k = 25 .. 33, color = [red, blue]);
```



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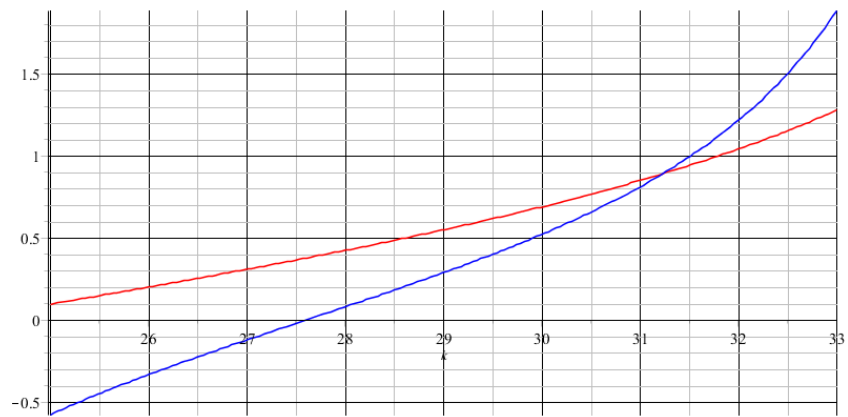
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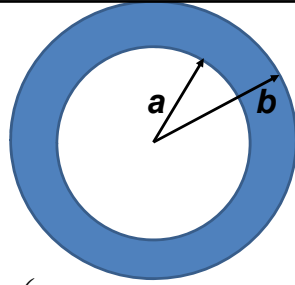
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Finding a solution graphically.

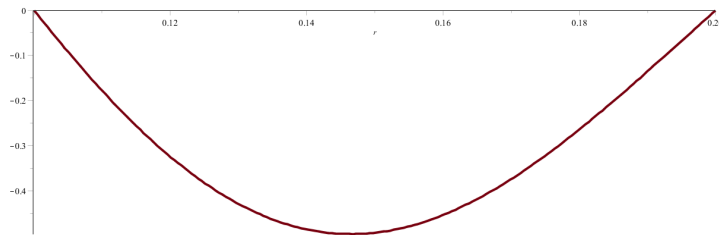
```
> fsolve( ( -BesselJ(0, 0.1*k) / BesselY(0, 0.1*k) = -BesselJ(0, 0.2*k) / BesselY(0, 0.2*k) ), k, 30..33);
```

31.23030920





$$f(r) = A \left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right) \quad k_{01} = 31.230309$$



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Solution for this case.