PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Discussion for Lecture 26 – Chaps. 5,7,8 (F & W)

Review

1. Moments of inertia and rigid body motion
2. Contour integration
3. Wave equation in one special dimension
4. Wave equation in two special dimensions
5. Special transcendental functions such as Bessel functions

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Jeanette – 11 AM Wednesday
Derek – 12 PM Friday

15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020
26	Fri, 10/23/2020	Chap. 5,7,8	Review	#17	10/28/2020
26	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids		

PHY 711 -- Assignment #17

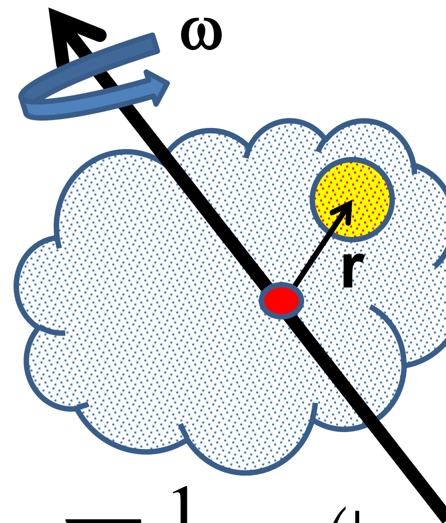
Oct. 23, 2020

Review Chapter 8 and Appendix E in **Fetter & Walecka**.

In Lecture 25, we considered the example of a circular membrane of radius *R*, whose wave motion is characterized by a wave velocity *c*, finding the normal mode frequencies ω_{mn} as multiples of *c/R*, for the case that circular boundary was clamped (zero displacement). That is, for a wave displacement function $\rho(r,\varphi)$ with $\rho(R,\varphi)=0$. Consider the same system for the case that the circular boundary is free which can be approximated by $\partial\rho(R,\varphi)/\partial r = 0$. List the three lowest frequencies (in units of *c/R*) compared with the clamped boundary values. Note that Fetter & Walecka lists zeros of Bessel functions in Appendix E4. These values can also be determined from Maple and Mathematica.

Rigid body motion – moments of inertia tensors

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

Kinetic energy: $$T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right) - \left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

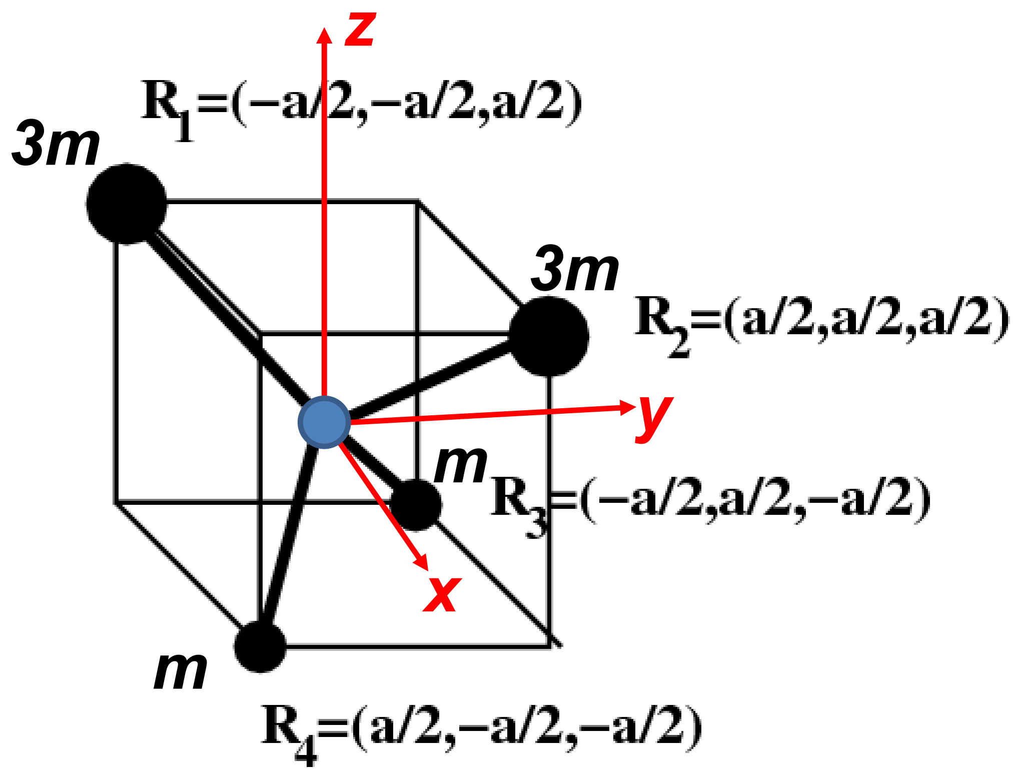
$$= \frac{1}{2}\boldsymbol{\omega}\cdot\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p\mathbf{r}_p\right)$$

Moment of inertia tensor

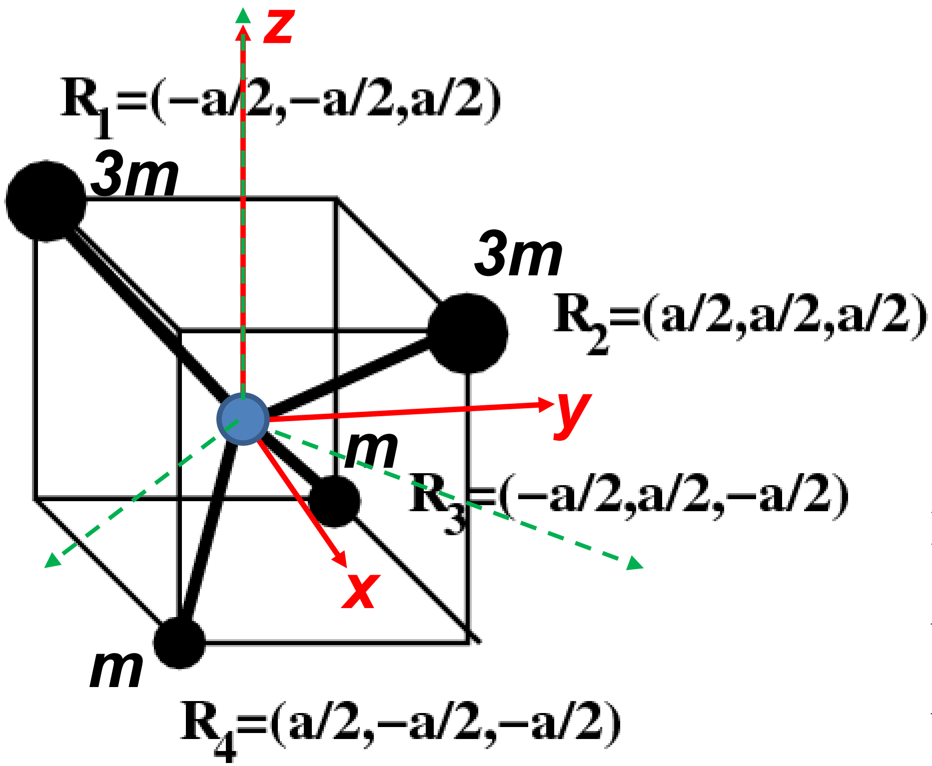
Matrix notation:

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Example of 5 atom molecule with origin at central atom.



$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$



$$I_{xx} = 4ma^2 = I_{yy} = I_{zz}$$

$$I_{xy} = -ma^2$$

$$\mathbf{I} = ma^2 \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Principal moments

$I_1 = 3ma^2$ with axis $\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + \hat{\mathbf{y}}\right)$

$I_2 = 3ma^2$ with axis $\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}}\left(-\hat{\mathbf{x}} + \hat{\mathbf{y}}\right)$

$I_3 = 4ma^2$ with axis $\hat{\mathbf{e}}_3 = \hat{\mathbf{z}}$

What happens when all masses are equal?

Is it an accident that the principal axes are related to the symmetry of the system?

Functions of complex variables,   contour integration

Contour integrals --

$$\oint_C f(z)dz = 2\pi i \sum_p \mathrm{Res}\left(f(z_p)\right)$$

General formula for determining residue for each pole z_p of order m

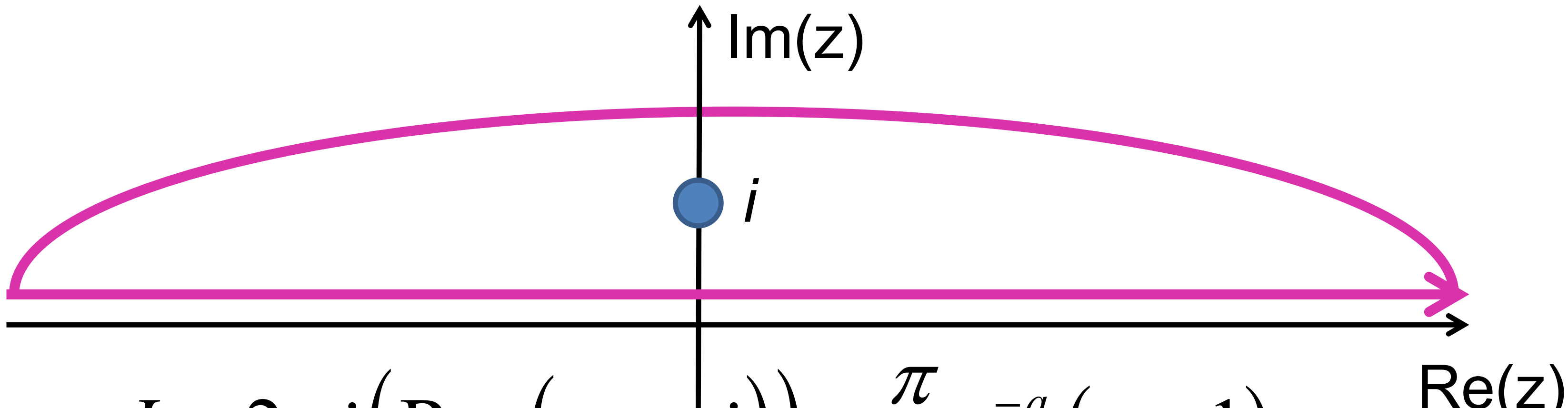
$$\mathrm{Res}\left(f(z_p)\right) = \lim_{z \to z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}} \right\}$$

Example --

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{\left(x^2+1\right)^2}dx = \oint \frac{e^{iaz}}{\left(z^2+1\right)^2}dz$$

For $a>0$

$$\left(z^2+1\right)^2 = \left(z-i\right)^2\left(z+i\right)^2$$



$$I = 2\pi i\left(\text{Res}\left(z_p = i\right)\right) = \frac{\pi}{2}e^{-a}\left(a+1\right)$$

Note that for $a>0$ only the pole at $z_p=i$ contributes and $m=2$

Note also that complex variables have other considerations such as “branch cuts” to avoid multiple values and essential singularities.

Wave motion in one special dimension

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

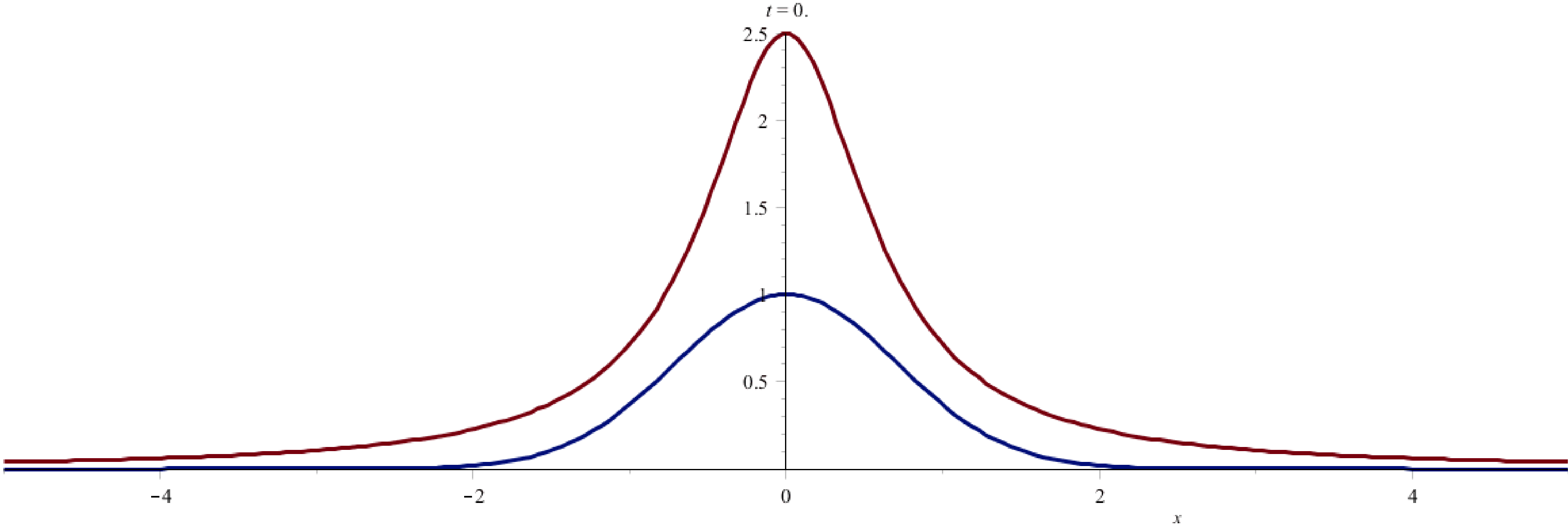
For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \dfrac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

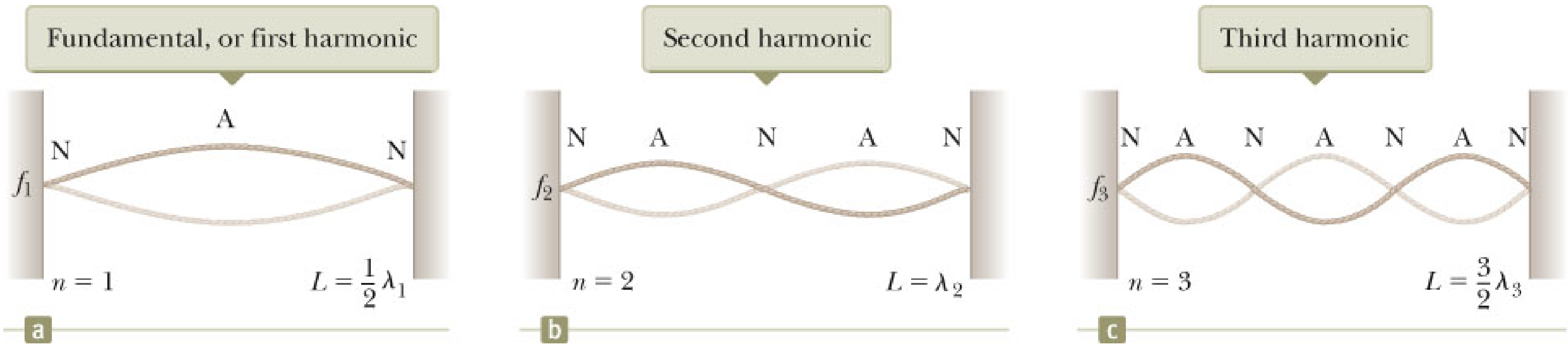
with $\mu(0,t) = \mu(L,t) = 0.$

Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

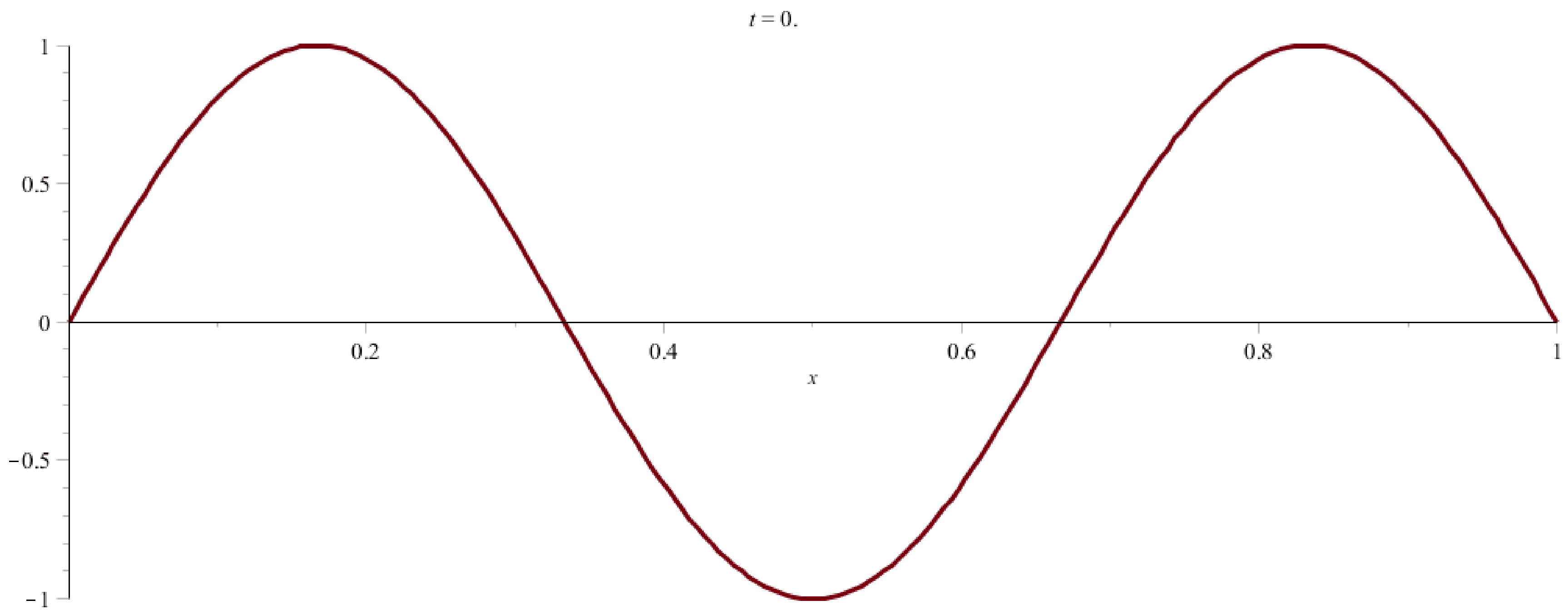
where $\frac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0$ $\qquad k = \frac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$



Fundamental, or first harmonic
A
N
N
f_1
$n = 1$
$L = \frac{1}{2}\lambda_1$
a
Second harmonic
N A N A N
f_2
$n = 2$
$L = \lambda_2$
b
Third harmonic
N A N A N A N
f_3
$n = 3$
$L = \frac{3}{2}\lambda_3$
c



t = 0.
1
0.5
0
−0.5
−1
0.2
0.4
0.6
0.8
1
x

Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

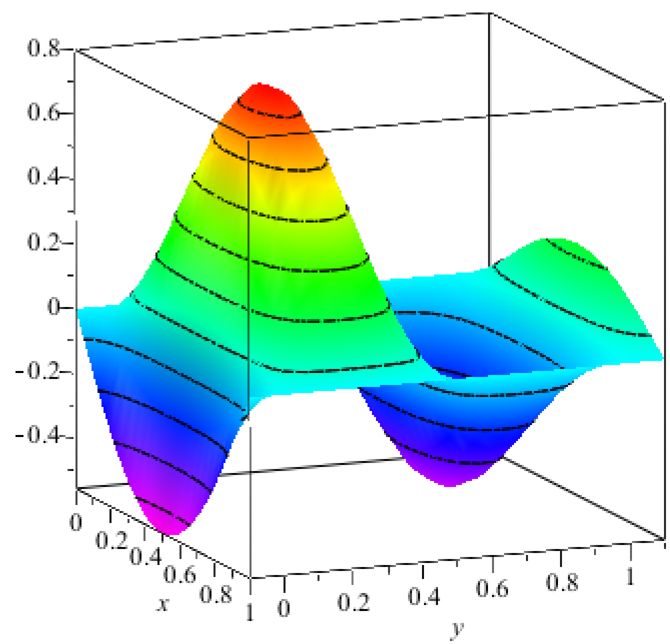
Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Note that here we are visualizing transverse waves. Longitudinal waves can also exist.

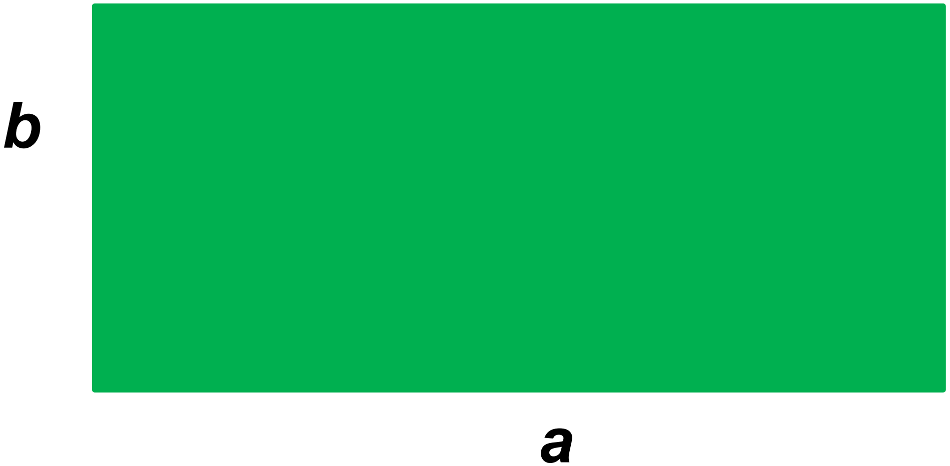
$\rho(x,y)$



In this case, we have mapped the one dimensional elastic string to a two dimensional elastic membrane

$$\frac{\partial^2}{\partial x^2} \rightarrow \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(in Cartesian coordinates)}$$

Consider a rectangular boundary:



Clamped boundary conditions :

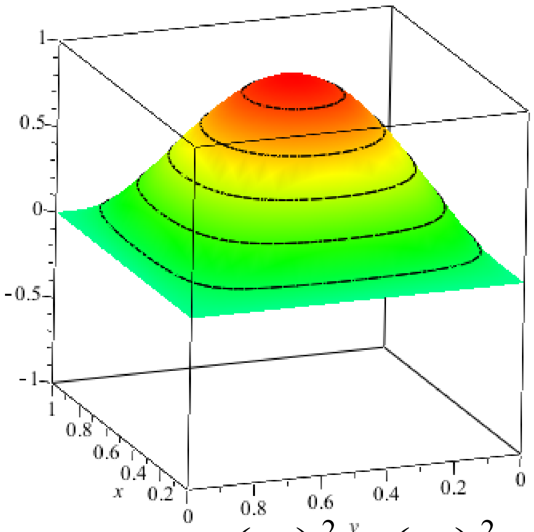
$$\rho(0,y)=\rho(a,y)=\rho(x,0)=\rho(x,b)=0$$

$$\Rightarrow \rho_{mn}(x,y)=A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

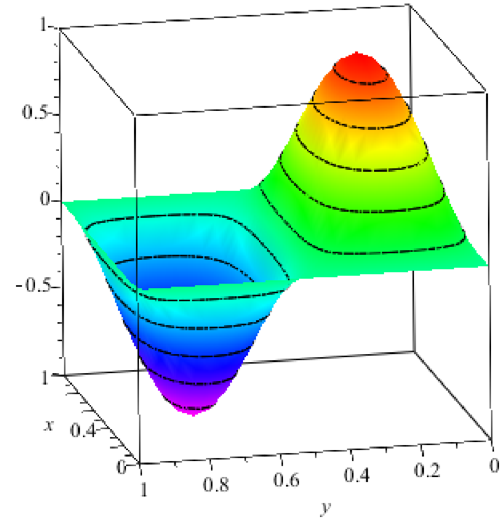
$$k_{mn}{}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$

$$\left(\nabla^2+k^2\right)\rho(x,y)=0$$

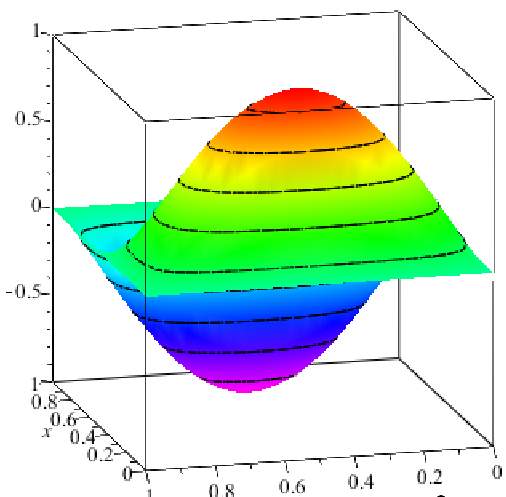
$$\text{where } k=\frac{\omega}{c}$$



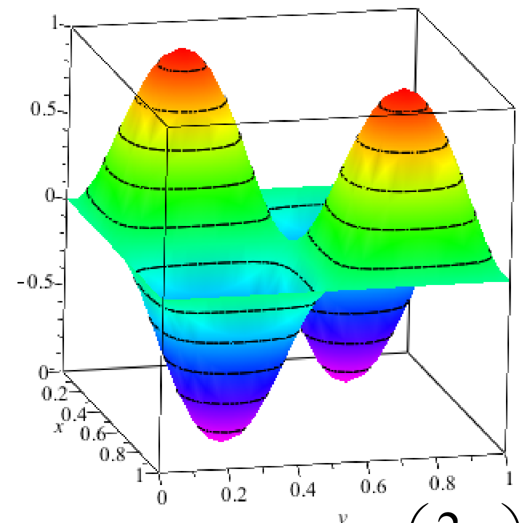
$$k_{11}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{12}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$



$$k_{21}^{\;2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{22}^{\;2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$\tau \nabla u\big|_b = \kappa u\big|_b$ represents bounded side constrained with spring

$\tau \nabla u\big|_b = 0$ represents "free" side

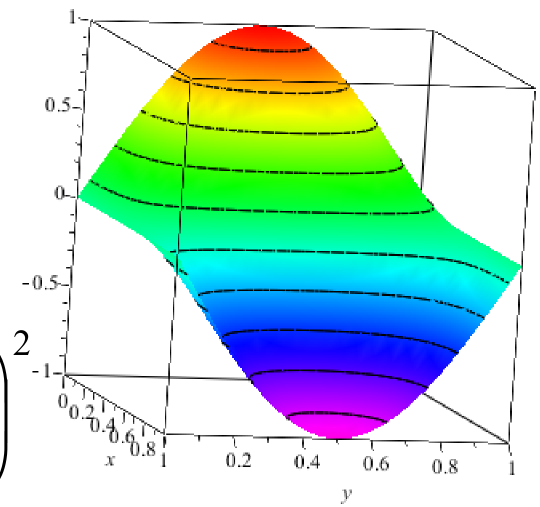
Mixed boundary conditions :

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

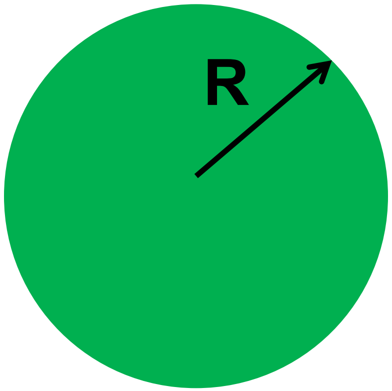
$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^{\ 2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$k_{11}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$$\rho(R,\varphi)=0$$

$$\left(\nabla^2+k^2\right)\rho(r,\phi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\Phi(\varphi)=e^{im\varphi}$

Note: $\Phi(\varphi)=\Phi(\varphi+2\pi)$

$$\Rightarrow m=\text{integer}$$

Consider circular boundary -- continued

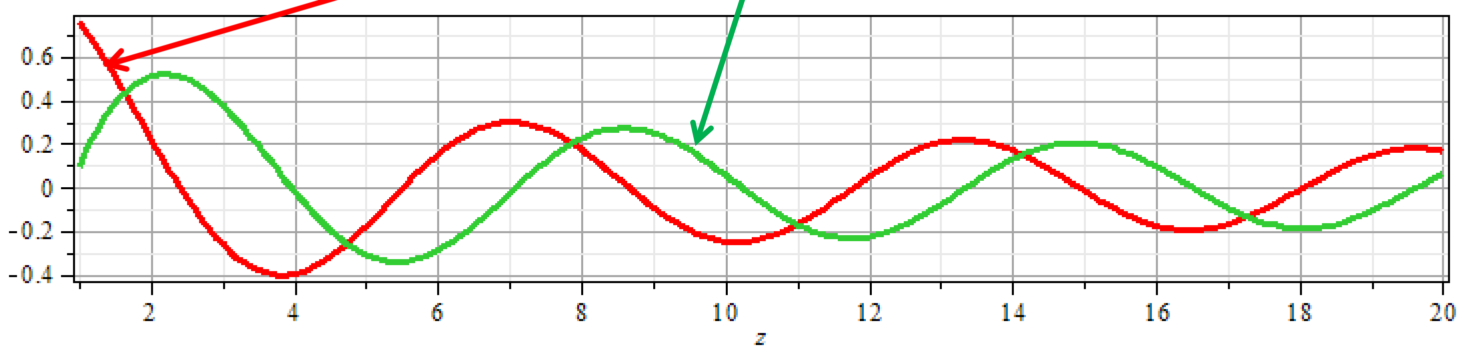
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



Some properties of Bessel functions

Asending series: $$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!} \left(\frac{z}{2}\right)^{2j}$$

Recursion relations: $$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$$

$$J_{m-1}(z) - J_{m+1}(z) = 2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $$J_m(z) \xrightarrow[z>>1]{} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Zeros of Bessel functions $J_m(z_{mn}) = 0$

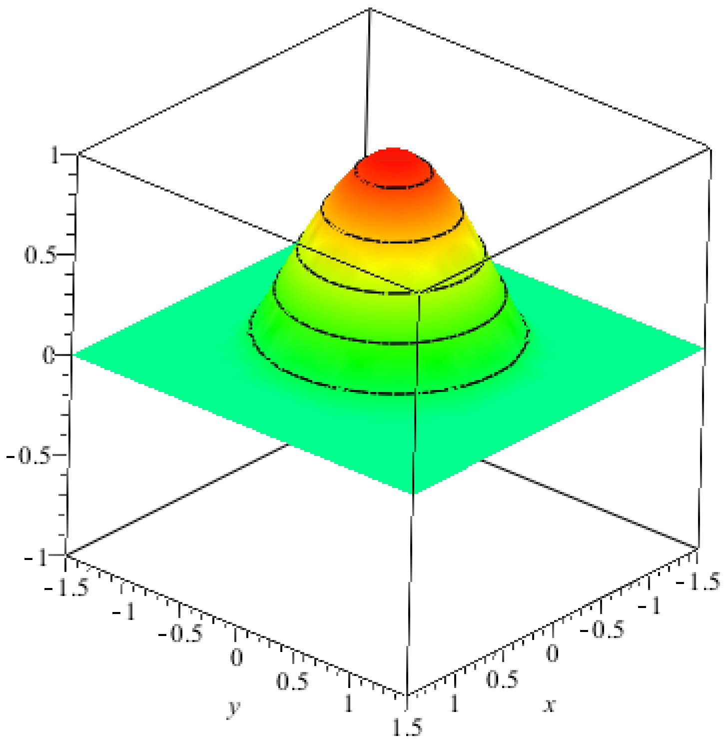
$m = 0: \quad z_{0n} = 2.406,\ 5.520,\ 8.654, \ldots$

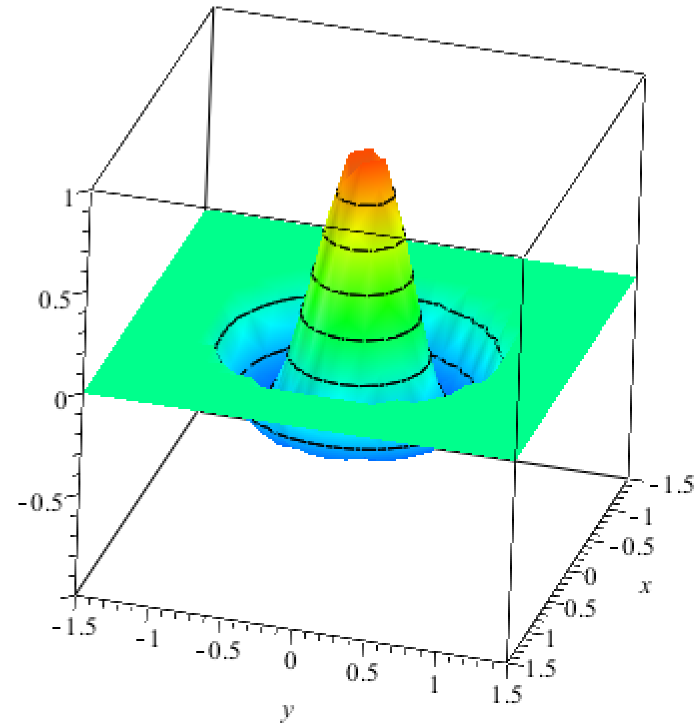
$m = 1: \quad z_{1n} = 3.832,\ 7.016,\ 10.173, \ldots$

$m = 2: \quad z_{2n} = 5.136,\ 8.417,\ 11.620, \ldots$

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right)$$

$$\rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$





$$k_{01} = \frac{2.406}{R}$$

$$k_{02} = \frac{5.520}{R}$$

Some details for *m=0*