PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Discussion for Lecture 27 – Chap. 9 in F & W

Introduction to hydrodynamics

1. Motivation for topic
2. Newton's laws for fluids
3. Conservation relations

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Jeanette – 11 AM Wednesday
Derek – 12 PM Friday

15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020
26	Fri, 10/23/2020	Chap. 5,7,8	Review	#17	10/28/2020
27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020

Homework #18

PHY 711 -- Assignment #18

Oct. 26, 2020

Read Chapter 9 in **Fetter & Walecka**.

1. Consider the example discussed in class on slides 20-24 concerning the flow of an incompressible fluid in the **z** direction in the presence of a stationary cylindrical log oriented in the **y** direction. For this problem, consider the case where the log is replaced by a stationary sphere. Find the velocity potential for this case, using the center of the sphere as the origin of the coordinate system and spherical polar coordinates.

Hydrodynamic analysis
Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Your questions –
From Tim

1. How come the pressure at point 1 on pg.14 includes p_atm? Is it because the atmosphere pushes on the syringe back, which then pushes on the fluid?

From Nick

1. What is the difference between F_applied and f_applied?
2. What is \Phi representing?
3. Is irrotational flow allowed to oscillate up and down or side to side, or pulse? but just not spin?

From Gao

1. What aspects do over simplified Bernoulli's equation not include in studying fluid dynamics?

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

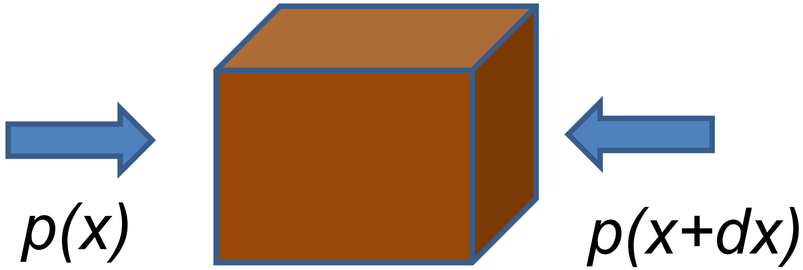
$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

$$m\mathbf{a} = \mathbf{F}$$

$$m \to \rho dV$$

$$\mathbf{a} \to \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \to \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \big(-p(x+dx,y,z)+p(x,y,z)\big)dydz$$

$$= \frac{\big(-p(x+dx,y,z)+p(x,y,z)\big)}{dx}dxdydz$$

$$= -\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$m = \rho dV$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\rho dV - (\nabla p) dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that : $\mathbf{v} \equiv v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

$$\frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho\left(\left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times \left(\nabla \times \mathbf{v} \right) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2} v^2 \right) - \mathbf{v} \times \left(\nabla \times \mathbf{v} \right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Your question – What is irrotational flow?

Irrotational flow: $\nabla \times \mathbf{v} = 0$

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Which of the following vector functions have zero curl?

a. $\mathbf{v}=C\hat{\mathbf{x}}$ (C is a constant)

b. $\mathbf{v}=Cx\hat{\mathbf{x}}$

c. $\mathbf{v}=Cy\hat{\mathbf{x}}$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla \times \mathbf{v}\right) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla\Phi \quad \Phi \text{ is "velocity potential"}$$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C'(t)\big)$

$$\Rightarrow \frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0$$

Modified form; assuming $\frac{\partial \Phi}{\partial t} = 0$

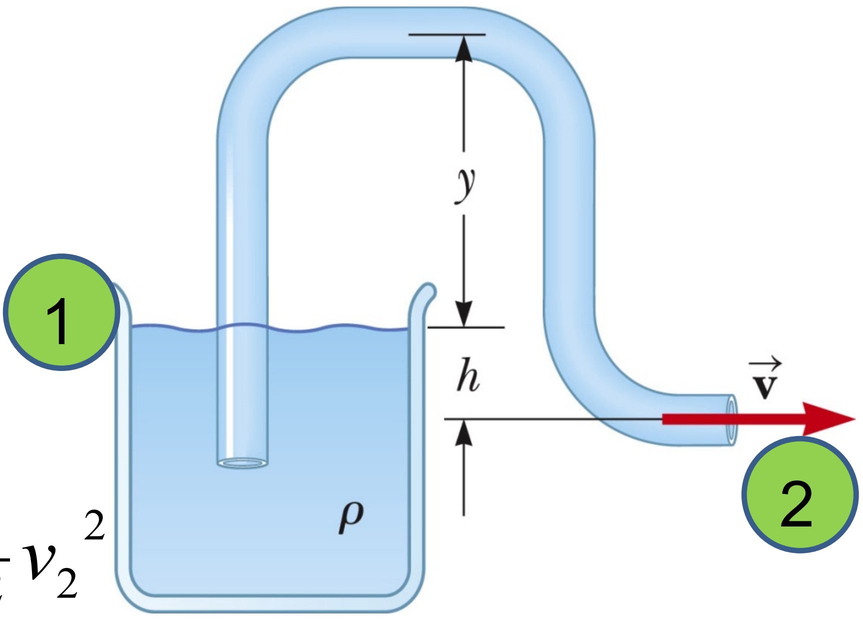
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

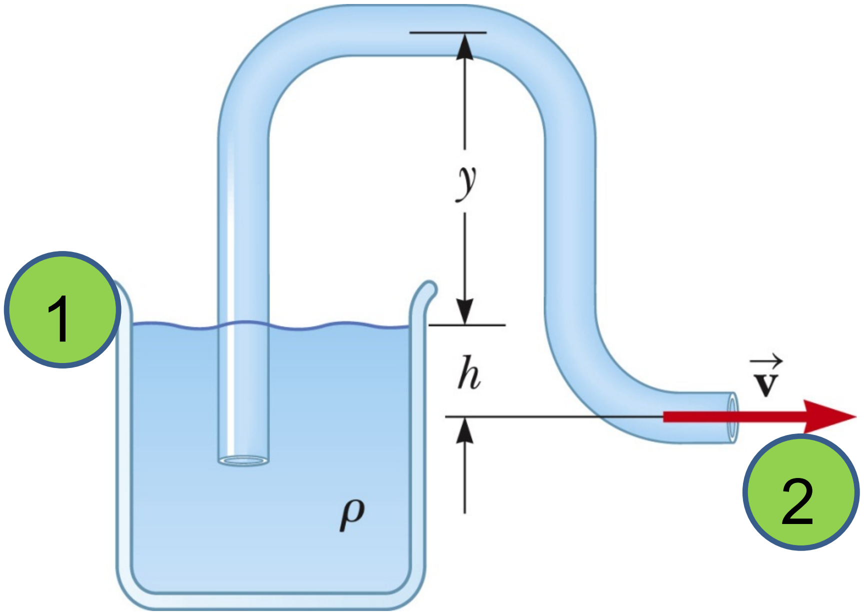
$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\,2}$$



Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

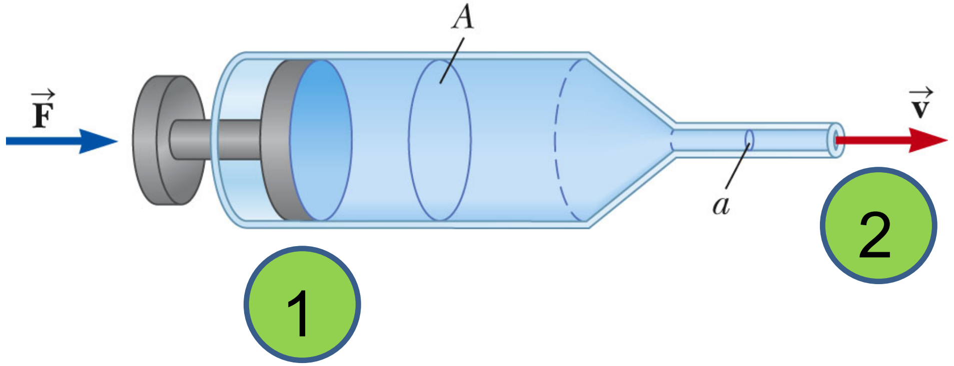
$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\ 2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\ 2}$$

$$v_2 \approx \sqrt{2gh}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \qquad p_2 = p_{atm}$$

$$U_1 = U_2$$

$$v_1 A = v_2 a \qquad \text{continuity equation}$$

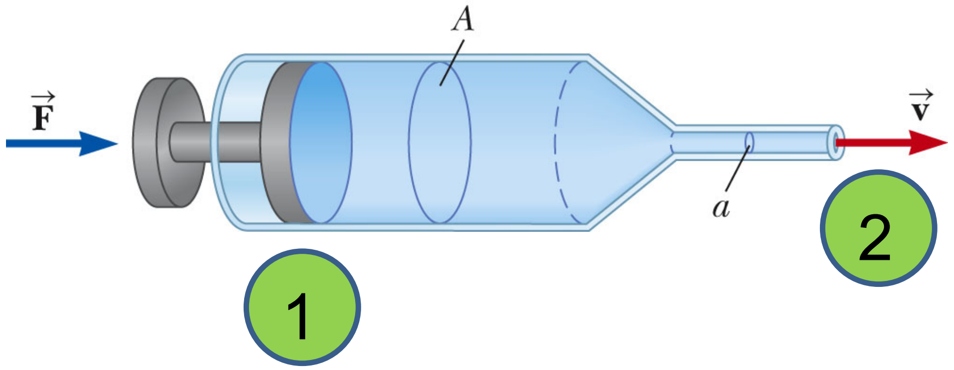
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

Your question -- How come the pressure at point 1 on pg.14 includes p_atm? Is it because the atmosphere pushes on the syringe back, which then pushes on the fluid?

Comment – All surfaces open to the air is in equilibrium with the atmospheric pressure.

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^{\,2}\left(1 - \left(\frac{a}{A}\right)^2\right)$$

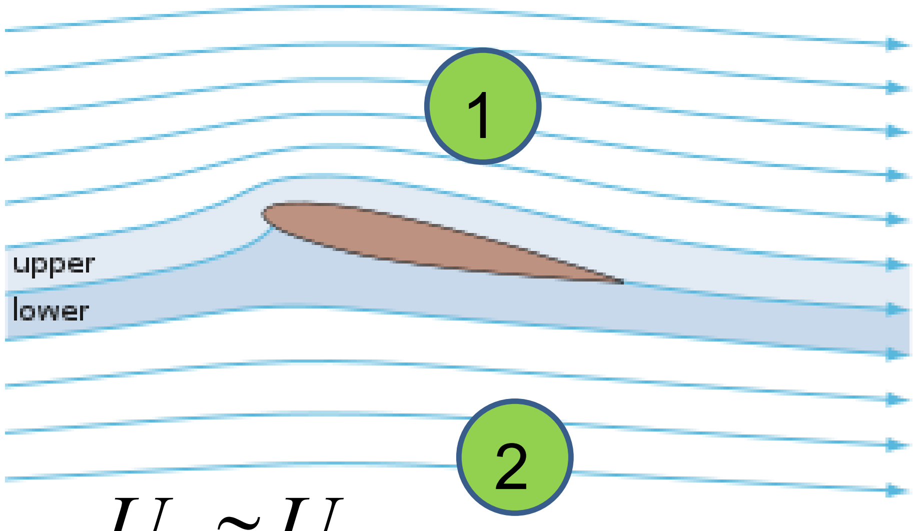
$$v_2 = \sqrt{\frac{2F/A}{1-\left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

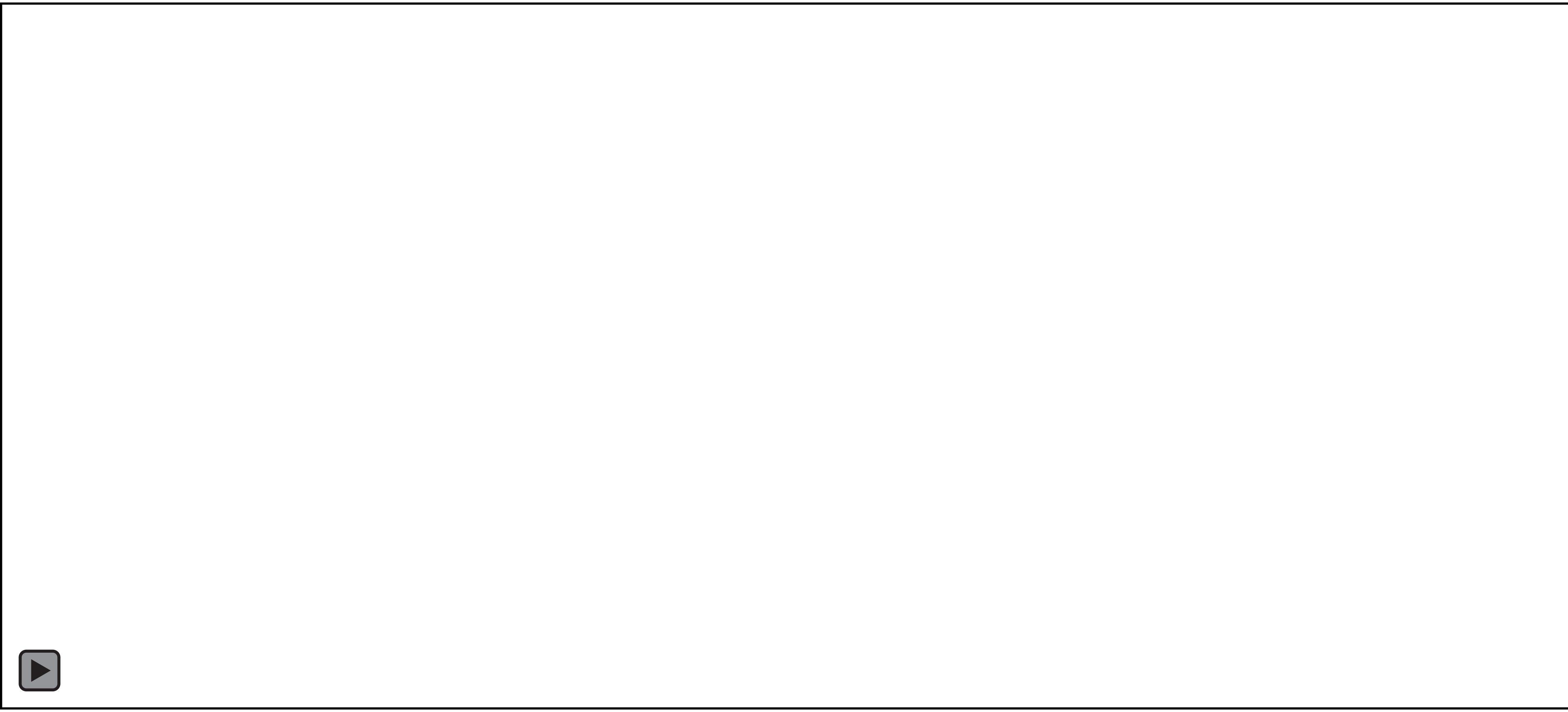
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2} v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2} v_2^{\,2}$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^{\,2} - v_2^{\,2}\right)$$

Your question -- What aspects do over simplified Bernoulli's equation not include in studying fluid dynamics?

According to a Scientific American article, the conclusion that $v_2>v_1$ because of the shape of the airplane wing is not quite true. Numerical modeling reveal a more complicated picture.

https://www.scientificamerican.com/article/no-one-can-explain-why-planes-stay-in-the-air/



At NASA Ames Fluid Mechanics Laboratory, streamlines of dye in a water channel interact with a model airplane. Credit: *Ian Allen* (copied from Scientific American page mentioned above).

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $\dfrac{d\rho}{dt} = \dfrac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$$\Rightarrow \frac{d\rho}{dt} + \rho (\nabla \cdot \mathbf{v}) = 0$$

alternative form

of continuity equation

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

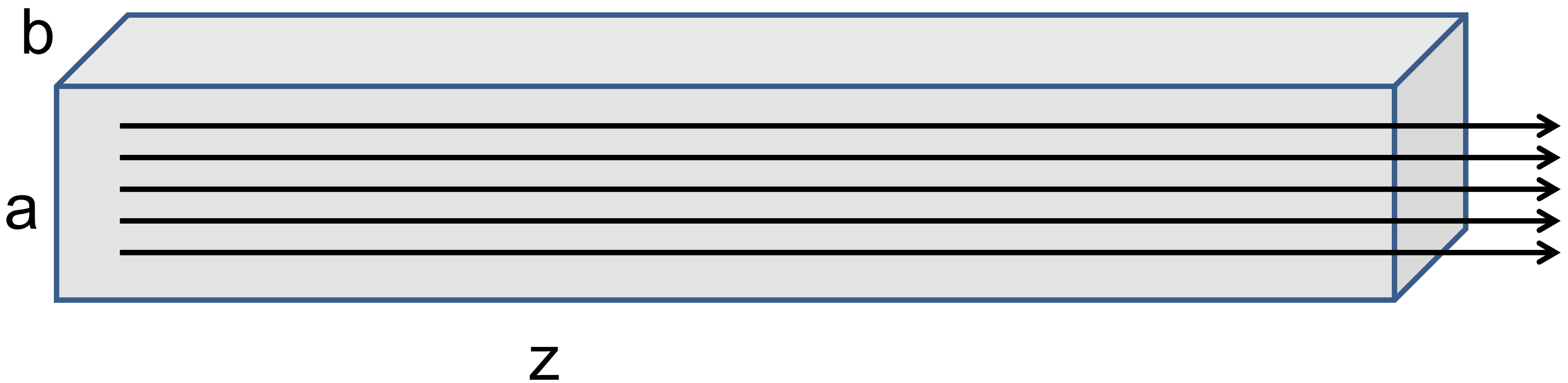
For incompressible fluid : $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow : $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

Example – uniform flow



$$\nabla^2\Phi = 0$$

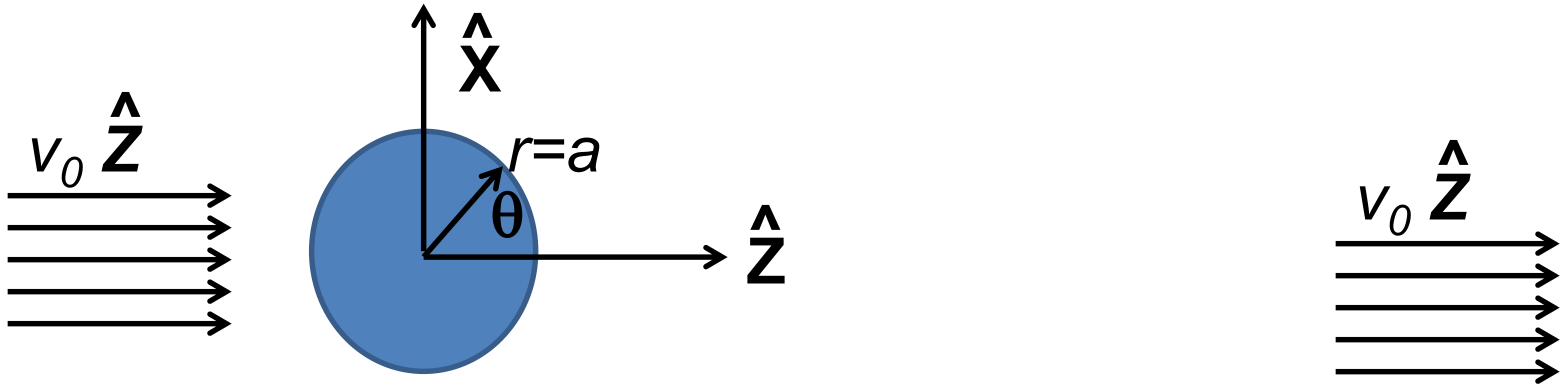
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the *Y* direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r,\theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r\to\infty) = v_0$

$$\frac{\partial\Phi}{\partial z}(r\to\infty) = -v_0 \qquad \Rightarrow \Phi(r\to\infty,\theta) = -v_0 r\cos\theta$$

Note that : $\frac{\partial^2\cos\theta}{\partial\theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}-\frac{1}{r^2}f=0$$

$$f(r)=Ar+\frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a}=0$$

$$\frac{df}{dr}(r=a)=0=A-\frac{B}{a^2}$$

$$\Rightarrow B=Aa^2$$

Boundary condition at ∞: $\quad \Rightarrow A=-v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

to be continued …