PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 27

Chap. 9 in F & W: Introduction to hydrodynamics
1. Motivation for topic
2. Newton's laws for fluids
3. Conservation relations

In this lecture we will begin an introductory treatment of the mechanics of fluis.

15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020
26	Fri, 10/23/2020	Chap. 5,7,8	Review	#17	10/28/2020
27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020

There is a homework problem based on today's lecture material

Homework #18

PHY 711 -- Assignment #18

Oct. 26, 2020

Read Chapter 9 in **Fetter & Walecka**.

1. Consider the example discussed in class on slides 20-24 concerning the flow of an incompressible fluid in the **z** direction in the presence of a stationary cylindrical log oriented in the **y** direction. For this problem, consider the case where the log is replaced by a stationary sphere. Find the velocity potential for this case, using the center of the sphere as the origin of the coordinate system and spherical polar coordinates.

This problem involves a description of simple fluid motion.

Hydrodynamic analysis
Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Here is a list of topics that will be covered in the next few lectures.

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

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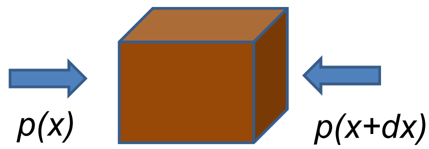
Newton's laws need to be adapted to describe the physics of fluids. Here pressure is important and more generally, the functions used to describe fluids depend on position and time.



$$F_{pressure}\Big|_x = \left(-p(x+dx,y,z)+p(x,y,z)\right)dydz$$

$$= \frac{\left(-p(x+dx,y,z)+p(x,y,z)\right)}{dx}dxdydz$$

$$= -\frac{\partial p}{\partial x}dV$$

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Pressure acts in all directions. Here we argue that the spatial derivative of the pressure applies a force to a volume of fluid.

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p)dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho\mathbf{f}_{applied} - \nabla p$$

It is convenient to write Newton's law in terms of the mass density, velocity, and pressure of the fluid.

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

Because of the continuous nature of the velocity, the total time derivative of the fluid velocity depends both or the partial derivates with respect to space and with respect to time as derived here.

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} \right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Some alternative expressions for the velocity terms.

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla \times \mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

The restricted equations have some interesting properties.

Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\left(\Phi(\mathbf{r},t)+C'(t)\right)$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

This result is known as Bernoulli's equation

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0$$

$$\text{Modified form; assuming } \frac{\partial \Phi}{\partial t} = 0$$

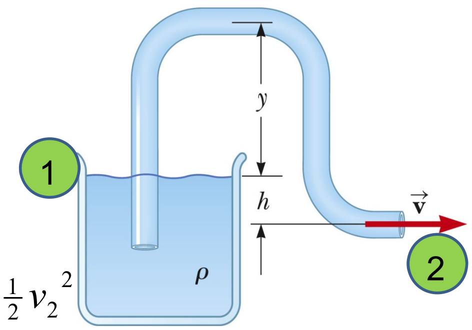
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

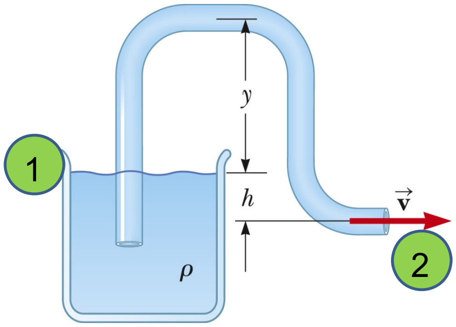
$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\,2}$$



This is a problem illustrating Bernoulli's equation as a syphon.

Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

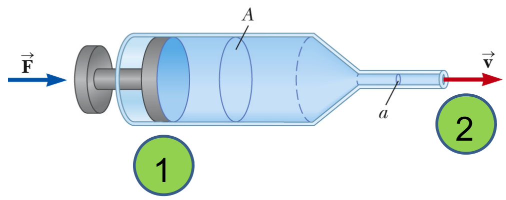
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2} v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2} v_2^{\,2}$$

$$v_2 \approx \sqrt{2gh}$$

This example is taken from the PHY 114 textbook

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \qquad p_2 = p_{atm}$$

$$U_1 = U_2$$

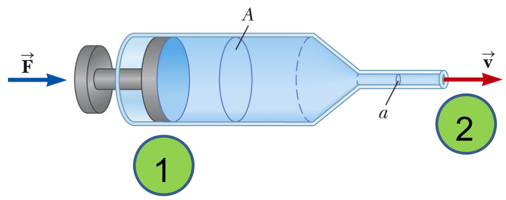
$$v_1 A = v_2 a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$

Another ezample of Bernoulli's equation for a syringe.

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2{}^2\left(1 - \left(\frac{a}{A}\right)^2\right)$$

$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

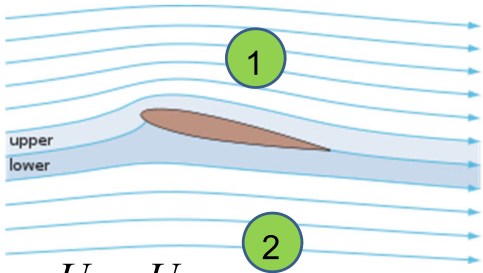
Syringe fluid continued.

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

This example of Bernoulli's equation is oversimplified. It appeared in most of the old textbook, but seems now to be deemphasized. It is given here since it shows some aspects of fluid flow, although apparently not good enough.

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0$ alternative form of continuity equation

The continuity equation is an important aspect of fluid flow.

Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

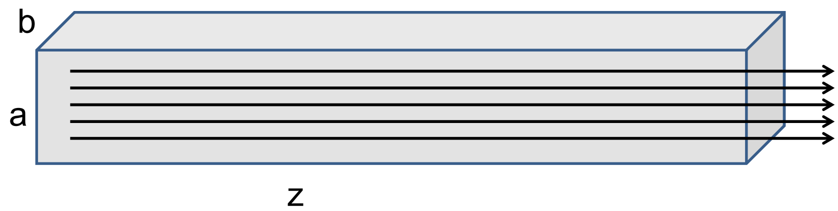
$\Rightarrow \nabla \cdot \mathbf{v} = 0$

Irrotational flow: $\nabla \times \mathbf{v} = 0$ $\qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$\qquad \Rightarrow \nabla^2 \Phi = 0$

For an incompressible and irrotational fluid, it is mathematically convenient to express the velocity field in terms of a velocity potential field.

Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

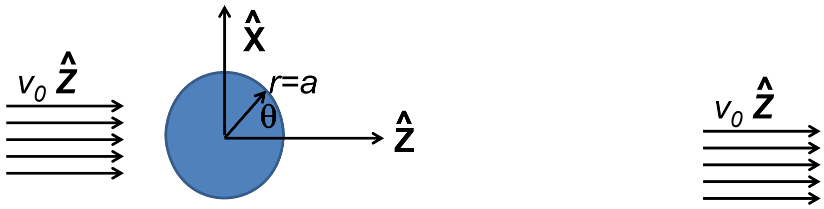
Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

For a uniformly fluid flowing along the z direction, the velocity potential and velocity field are easily written as shown.

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Now consider the uniform fluid in the presence of an impediment. In the is case we consider a cylindrical log.

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r, \theta) = f(r) \cos\theta$

We need to consider solutions of the Laplace equation.

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r = a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞: $\Rightarrow A = -v_0$

Particular equations for this geometry and the application of the boundary values.

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

to be continued …

More details.