PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Discussion on Lecture 27 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Irrotational and incompressible fluids
3. Irrotational and isentropic fluids
4. Approximate solutions in the linear limit – next time

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Tim – 11 AM Wednesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020
26	Fri, 10/23/2020	Chap. 5,7,8	Review	#17	10/28/2020
27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		

WAKE FOREST UNIVERSITY

Department of Physics

Colloquium

NEWEST FACULTY JOINT PRESENTATIONS

Thursday Oct. 29, 2020 at 4 PM

Ilaria Bargigia, PhD
Assistant Professor
Physics Department
Wake Forest University,
Winston-Salem, NC

"Organic Bio-Electronics for In-Vivo Applications"

Conjugated polymers are widely used as bio-electronic interfaces thanks to their inherent softness, biocompatibility, and unparalleled versatility. In particular, thin films of poly(3-exylthiophene) have demonstrated the capability to restore light sensitivity in animal models and are now being proposed as artificial retinal implants. However, there is no clear understanding of the mechanism behind light-induced activation of cellular activity mediated by the photophysical characteristics of the conjugated polymers: hence, there is a need to address how structural properties and the local environment control the various functionalities, and to investigate the role played by the interface between the polymer and biological media. In this talk, I will present our recent efforts made towards the understanding of how photo-physical properties transform in the presence of relevant biological media and what these transformations entail in the context of in-vivo biological applications. In particular, I will focus on the nature of electrochemically induced charges, and their coupling to the local environment.

Ajay Ram Srimath Kandada, PhD
Assistant Professor
Physics Department
Wake Forest University,
Winston-Salem, NC

"Optical Probes of System-Bath Interactions in Emerging Semiconductors"

Photo-excitation dynamics in condensed matter not only depend on the intrinsic properties of the material system but also on the interactions with the environment, termed as the bath. Experimental assessment of the system-bath interactions forms the core of material investigations and drives the development of optimal material architectures. Spectroscopies, especially those based on optical probes, predominantly involve identification of the consequences of such interactions and thereby their quantification. I will discuss the outstanding challenges in the current state-of-the art spectroscopic techniques used for this purpose. I will also present a roadmap for developing methodologies based on ultrashort optical pulses and quantum-entangled photons to overcome these challenges, especially in the context of emerging two-dimensional materials.

Steve Winter, PhD
Assistant Professor
Physics Department
Wake Forest University,
Winston-Salem, NC

"Towards Many-Body Extensions of Symmetry Protected Topological Invariants"

In recent years, a particularly prominent field of Quantum Materials research has focused on topological phases of weakly interacting electrons, such as topological insulators and semimetals. In such materials, due to the specific symmetries and connectivity of the electronic bands in reciprocal space, the bands cannot be smoothly deformed to the trivial atomic limit without closing energy gaps or breaking symmetries. This can lead, for example, to additional gapless states near crystal edges, which are partially protected against scattering, thus providing very high electronic mobilities. Impressively, a complete classification of non-interacting phases in all 230 space groups is already given by the topological quantum chemistry (TQC) formalism [1], which links symmetry properties of states in r- and k-space. The question that we address here is whether such a formalism also exists for interacting phases where electron-electron interactions are strong? We argue that direct application of the TQC formalism to general many-body spectra is impossible, but a complete classification of the (large) subset of states connectable to interacting atomic limits can be achieved via extended TQC approaches.

[1] B. Bradlyn et al. Nature, 547(7663), 298 (2017).

Your questions –
From Tim

1. Is there any physical relationship with the variable epsilon on slide 19?

From Nick

1. Can you spend some time going over the big picture of the continuity equation, and where it comes from and what is means? Are we just working with an example of a continuity equation here, or is this same one applied elsewhere?

From Gao

1. About today's lecture, Why do zero velocity curl lead to velocity potential?

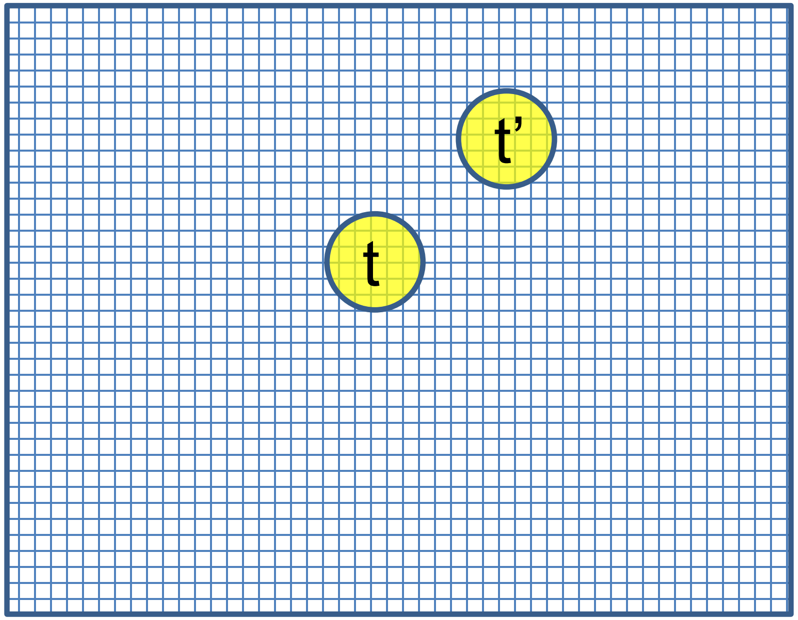
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$



$$\text{Particle at } t: \quad \mathbf{r}, t$$

$$\text{Particle at } t': \quad \mathbf{r} + \mathbf{v}\,\delta t, t'$$

$$t' = t + \delta t$$

Euler analysis -- continued

$$\text{Particle at } t: \quad \mathbf{r}, t$$

$$\text{Particle at } t': \quad \mathbf{r} + \mathbf{v}\delta t, t' \qquad \text{where} \quad \delta t = t' - t$$

$$\text{For} \quad f(\mathbf{r}, t):$$

$$\frac{df}{dt} = \lim_{\delta t \to 0} \left(\frac{f(\mathbf{r}, t') - f(\mathbf{r}, t)}{\delta t} + \frac{f(\mathbf{r} + \mathbf{v}\delta t, t) - f(\mathbf{r}, t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

It can be shown that: $(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v})$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

For $f \rightarrow v_x$ $\quad \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + (\mathbf{v} \cdot \nabla) v_x$

For $f \rightarrow v_y$ $\quad \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + (\mathbf{v} \cdot \nabla) v_y$

For $f \rightarrow v_z$ $\quad \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + (\mathbf{v} \cdot \nabla) v_z$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}$

Note that $(\mathbf{v} \cdot \nabla) \mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) \left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right)$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla\right)\mathbf{v}$

Note that $\left(\mathbf{v} \cdot \nabla\right)\mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right)\left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right)$

$$= \frac{1}{2} \nabla \left|\mathbf{v}\right|^2 - \mathbf{v} \times \left(\nabla \times \mathbf{v} \right)$$

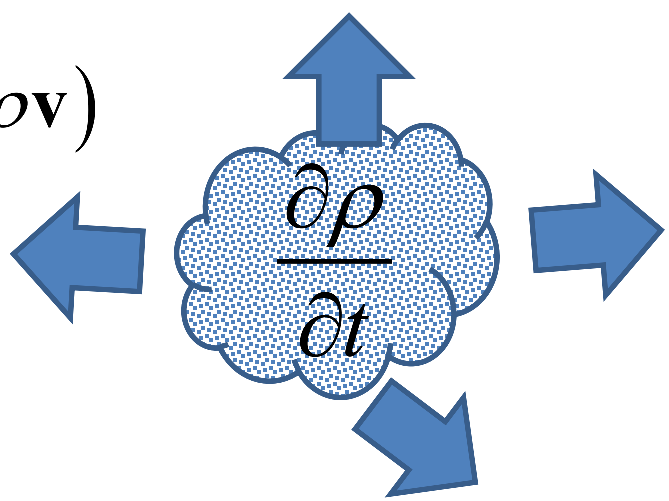
Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

The notion of the continuity is a common feature of continuous closed systems. Here we assume that there are no mechanisms for creation or destruction of the fluid.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$



$$\nabla \cdot (\rho \mathbf{v})$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

velocity potential

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow: $\nabla \times \mathbf{v} = 0$ $\quad \Rightarrow \mathbf{v} = -\nabla \Phi$

For irrotational flow of an incompressible fluid: $\nabla^2 \Phi = 0$

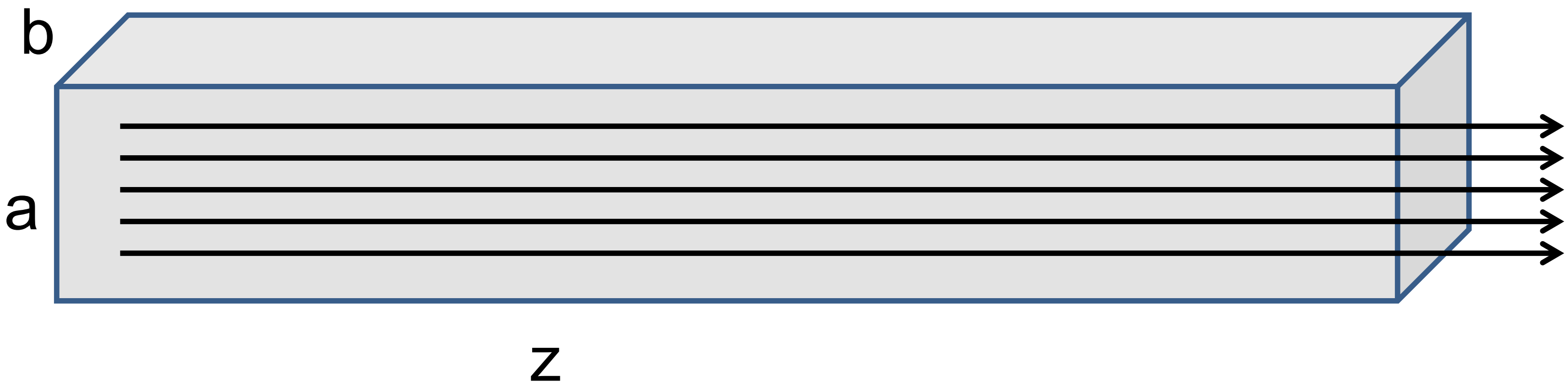
Your question

Why does $\nabla \times \mathbf{v} = 0$ imply that $\mathbf{v} = -\nabla\Phi$?

Consider: $\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}$

$\nabla \times (\nabla\Phi)\Big|_x = \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial y} = 0$ Similar results for other directions.

Example – uniform flow



$$\nabla^2\Phi = 0$$

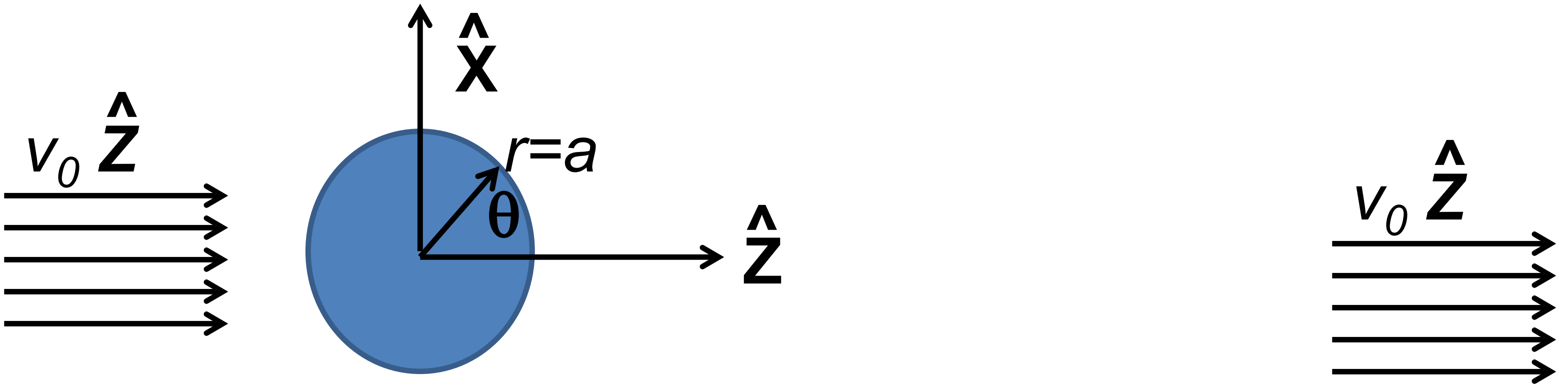
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the *Y* direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

(r, θ, defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition : $v_z(r \rightarrow \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \rightarrow \infty) = -v_0 \qquad \Rightarrow \Phi(r \rightarrow \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r, \theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}-\frac{1}{r^2}f=0$$

$$f(r)=Ar+\frac{B}{r}$$ where A, B are constants

Boundary condition on cylinder surface :

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a}=0$$

$$\frac{df}{dr}(r=a)=0=A-\frac{B}{a^2}$$

$$\Rightarrow B=Aa^2$$

Boundary condition at ∞ : $\Rightarrow A=-v_0$

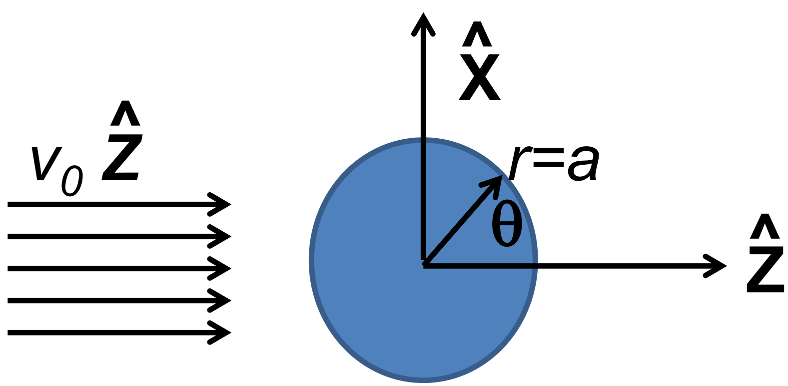
$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

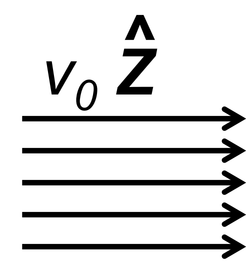
$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For $r \to \infty$

$$\mathbf{v} \to v_0 \cos\theta\,\hat{\mathbf{r}} - v_0 \sin\theta\,\hat{\boldsymbol{\theta}} = v_0\hat{\mathbf{z}}$$





Now consider the case of your homework problem --

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla \times \mathbf{v}\right) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla \Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t}\right) = 0$$

For incompressible fluid

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\left(\Phi(\mathbf{r},t)+C(t)\right)$

It is convenient to modify $\Phi(\mathbf{r},t)\rightarrow\Phi(\mathbf{r},t)+\int^{t}C(t')dt'$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Not all fluids are compressible, but with additional work we can consider fluids at constant entropy (no heat transfer).

Under what circumstances can there be no heat transfer?

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right)=0$ "irrotational flow"

$$\Rightarrow \mathbf{v}=-\nabla\Phi$$

2. $\mathbf{f}_{applied}=-\nabla U$ conservative applied force

3. $\rho\neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}}=dQ-dW$

For isentropic conditions: $dQ=0$

$$dE_{\text{int}}=-dW=-pdV$$

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -p\,dV$$

In terms of mass density: $\rho = \dfrac{M}{V}$

For fixed M and variable V: $d\rho = -\dfrac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = M d\varepsilon = -dW = -p\,dV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Is this useful?

a. Yes

b. No

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.