PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 27 -- Chap. 9 in F & W

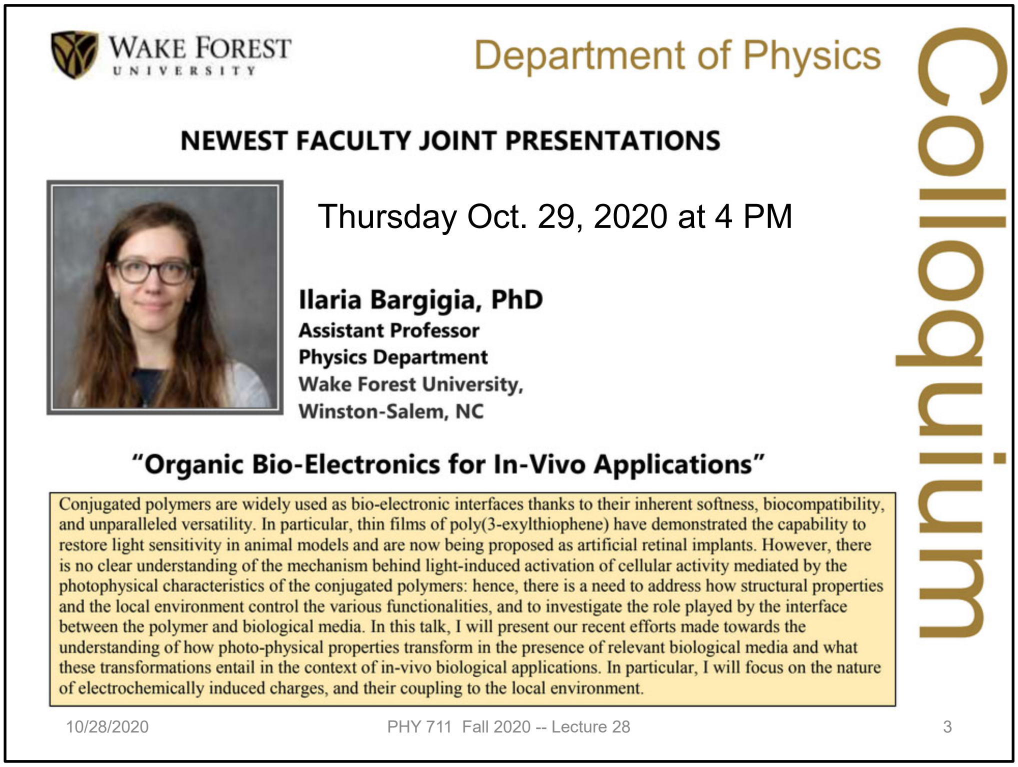
Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Irrotational and incompressible fluids
3. Irrotational and isentropic fluids
4. Approximate solutions in the linear limit

In this lecture, we will continue our discussion of hydrodynamics which is presented in Chapter 9 of your textbook.

15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020
26	Fri, 10/23/2020	Chap. 5,7,8	Review	#17	10/28/2020
27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		

Homework 18 is due Friday.



Our colloquium for this week features our three newest faculty members presenting snapshots of their research programs.

Ajay Ram Srimath Kandada, PhD
Assistant Professor
Physics Department
Wake Forest University,
Winston-Salem, NC

"Optical Probes of System-Bath Interactions in Emerging Semiconductors"

Photo-excitation dynamics in condensed matter not only depend on the intrinsic properties of the material system but also on the interactions with the environment, termed as the bath. Experimental assessment of the system-bath interactions forms the core of material investigations and drives the development of optimal material architectures. Spectroscopies, especially those based on optical probes, predominantly involve identification of the consequences of such interactions and thereby their quantification. I will discuss the outstanding challenges in the current state-of-the art spectroscopic techniques used for this purpose. I will also present a roadmap for developing methodologies based on ultrashort optical pulses and quantum-entangled photons to overcome these challenges, especially in the context of emerging two-dimensional materials.

Steve Winter, PhD
Assistant Professor
Physics Department
Wake Forest University,
Winston-Salem, NC

"Towards Many-Body Extensions of Symmetry Protected Topological Invariants"

In recent years, a particularly prominent field of Quantum Materials research has focused on topological phases of weakly interacting electrons, such as topological insulators and semimetals. In such materials, due to the specific symmetries and connectivity of the electronic bands in reciprocal space, the bands cannot be smoothly deformed to the trivial atomic limit without closing energy gaps or breaking symmetries. This can lead, for example, to additional gapless states near crystal edges, which are partially protected against scattering, thus providing very high electronic mobilities. Impressively, a complete classification of non-interacting phases in all 230 space groups is already given by the topological quantum chemistry (TQC) formalism [1], which links symmetry properties of states in r- and k-space. The question that we address here is whether such a formalism also exists for interacting phases where electron-electron interactions are strong? We argue that direct application of the TQC formalism to general many-body spectra is impossible, but a complete classification of the (large) subset of states connectable to interacting atomic limits can be achieved via extended TQC approaches.

[1] B. Bradlyn et al. Nature, 547(7663), 298 (2017).

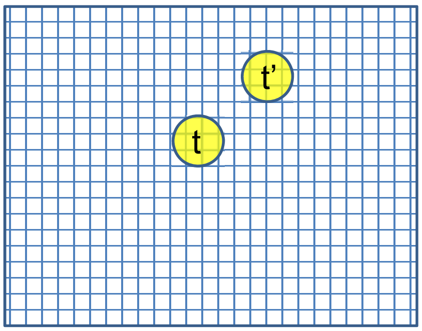
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$



Particle at t: $\mathbf{r}, t$

Particle at t': $\mathbf{r} + \mathbf{v}\delta t, t'$

$$t' = t + \delta t$$

Resuming our discussion of Newton's equations for fluids. For reference, this approach is named for Euler and is based on the continuous fluid being represented within an infinitesimal volume.

Euler analysis -- continued

$$\text{Particle at } t: \quad \mathbf{r}, t$$

$$\text{Particle at } t': \quad \mathbf{r}+\mathbf{v}\delta t, t' \qquad \text{where} \quad \delta t = t' - t$$

$$\text{For } \ f(\mathbf{r},t):$$

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{f(\mathbf{r},t') - f(\mathbf{r},t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t, t) - f(\mathbf{r},t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left(\mathbf{v}\cdot\nabla\right) f$$

Example: $\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right)$

While the infinitesimal volume moves from t to t’, the spatial position moves from r to r+v δt

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

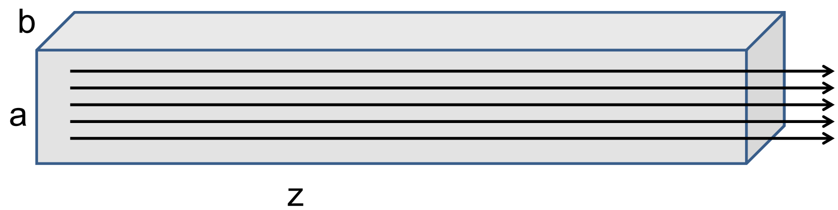
velocity potential

Irrotational flow: $\nabla \times \mathbf{v} = 0$ $\Rightarrow \mathbf{v} = -\nabla \Phi$

For irrotational flow of an incompressible fluid: $\nabla^2 \Phi = 0$

Another aspect of the fluid is the continuity equation. This simplifies to a velocity field which has zero divergence.
For irrotational flow the velocity field has zero curl and therefore can be written in terms of the velocity potential. Irrotational flow of an incompressible fluid satisfies the Laplace equation.

Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

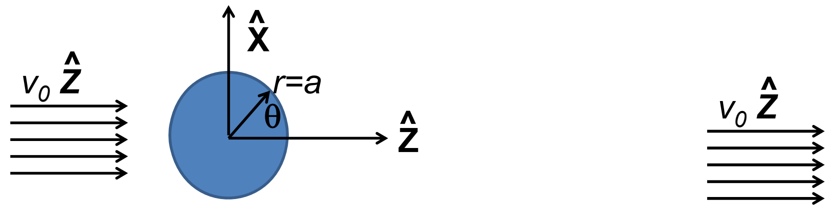
Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

Consider an example of irrotational flow of an incompressible fluid. In this case the fluid is flowing uniformly along the z axis.

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

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Now imagine the there is a log that distorts the flow. Here the long axis of the log is in the direction perpendicular to the screen. At the boundary of the log, the radial velocity is 0.

Laplace equation in cylindrical coordinates

($r,\theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r\to\infty) = v_0$

$$\frac{\partial\Phi}{\partial z}(r\to\infty) = -v_0 \qquad \Rightarrow \Phi(r\to\infty,\theta) = -v_0 r\cos\theta$$

Note that : $\frac{\partial^2\cos\theta}{\partial\theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Setting up and solving the boundary value problem.

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}-\frac{1}{r^2}f=0$$

$$f(r)=Ar+\frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a}=0$$

$$\frac{df}{dr}(r=a)=0=A-\frac{B}{a^2}$$

$$\Rightarrow B=Aa^2$$

Boundary condition at ∞: $\Rightarrow A=-v_0$

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Some details.

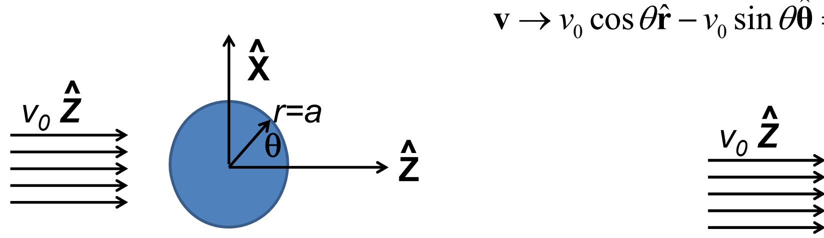
$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For $r \to \infty$

$$\mathbf{v} \to v_0 \cos\theta \hat{\mathbf{r}} - v_0 \sin\theta \hat{\boldsymbol{\theta}} = v_0 \hat{\mathbf{z}}$$



Full solution and simplified behavior far from the log.

Now consider the case of your homework problem --

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Consider a more complicated situation where there is a pressure gradient and applied potential. Specializing to the case of irrotational flow and arriving at the Bernoulli equation.

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C(t)\big)$

$$\Rightarrow \frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C_0$$

Bernoulli's theorem

For incompressible fluid

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla \times \mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics : $dE_{\text{int}} = dQ - dW$

For isentropic conditions : $dQ = 0$

$dE_{\text{int}} = -dW = -pdV$

Now consider generalizing this result to a possibly compressible fluid under the condition of zero heat transfer (isentropic).

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -pdV$$

In terms of mass density: $\rho = \frac{M}{V}$

For fixed M and variable V: $d\rho = -\frac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Here we need to introduce the so called first law of thermodynamics. This condition finds a general expression for ratio of the pressure and density in terms of the density derivative of the internal energy density.

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Is this useful?

a. Yes
b. No

This can be rearranged in terms of the gradient of the pressure divided by the density.

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Finally we arrive at a Bernoulli relation for irrotational flow of an isentropic material.

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Summary of results.