PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 29 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Approximate solutions in the linear limit

In this lecture, we will continue our discussion of hydrodynamics which is presented in Chapter 9 of your textbook. The focus will be on treating the equations in the linear regime.

15	Mon, 9/28/2020	Chap. 4	Small oscillations about equilibrium	#11	10/02/2020
16	Wed, 9/30/2020	Chap. 4	Normal modes of vibration	#12	10/05/2020
17	Fri, 10/02/2020	Chap. 4	Normal modes of vibration		
18	Mon, 10/05/2020	Chap. 7	Motion of strings	#13	10/07/2020
19	Wed, 10/07/2020	Chap. 7	Sturm-Liouville equations	#14	10/09/2020
20	Fri, 10/09/2020	Chap. 7	Sturm-Liouville equations		
21	Mon, 10/12/2020	Chap. 7	Fourier transforms and Laplace transforms		
22	Wed, 10/14/2020	Chap. 7	Complex variables and contour integration		
23	Fri, 10/16/2020	Chap. 5	Rigid body motion		
24	Mon, 10/19/2020	Chap. 5	Rigid body motion	#15	10/21/2020
25	Wed, 10/21/2020	Chap. 8	Elastic two-dimensional membranes	#16	10/23/2020
26	Fri, 10/23/2020	Chap. 5,7,8	Review	#17	10/28/2020
27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020

Updated schedule

PHY 711 -- Assignment #19

Oct. 28, 2020

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Using the analysis covered in class, estimate the speed of sound in the fluid of He gas at 1 atmosphere of pressue and at 300K temperature.

Recall the basic equations of hydrodynamics

Basic variables: Density $\rho(\mathbf{r},t)$

Velocity $\mathbf{v}(\mathbf{r},t)$

Pressure $p(\mathbf{r},t)$

Newton-Euler equation of motion:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

+ relationships among the variables due to principles of thermodynamics due to the particular fluid (In fact, we will focus on an ideal gas.)

Review of the basic equations of hydrodynamics.

Now consider the fluid to be air near equilibrium

Near equilibrium:

$\rho = \rho_0 + \delta\rho$ ρ_0 represents the average air density

$p = p_0 + \delta p$ p_0 represents the average air pressure

$\mathbf{v} = 0 + \delta\mathbf{v}$ (usually $\approx$ 1 atmosphere)

$\mathbf{f}_{applied} = 0$ $\mathbf{v}_0 = 0$ average velocity

Now consider air as the fluid near equilibrium with small fluctuations represented by the delta notation

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \qquad \Rightarrow \qquad \frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \Rightarrow \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta \mathbf{v} = -\nabla \Phi$$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0$$

Coupled equations.

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s, \rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s, \rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho \qquad \text{Here} \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Decoupling the equations in the velocity potential and the density fluctuation.

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

The decoupled equation is a wave equation in the velocity potential, density fluctuation, and pressure fluctuation variables.

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas :

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38 \times 10^{-23} J/k$$

M_0 = average mass of each molecule

10/28/2020 PHY 711 Fall 2020 -- Lecture 28 9

Estimating the wave velocity for air assuming that it is an ideal gas.

Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Using the ideal gas law with f representing the degrees of freedom. It is convenient to replace the f with the gamma factor which can be measured experimentally.

Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma-1} NkT \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Using the ideal gas law under isentropic conditions to derive the speed of sound.

Alternative derivation:

Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0^{\ 2} = \left(\frac{\partial p}{\partial \rho}\right)_{s, p_0, \rho_0} = \frac{p_0 \gamma}{\rho_0}$$

$$c_0^{\ 2} \approx \frac{1.5 \cdot 1.013 \times 10^5 \, Pa}{1.3 kg / m^3} \qquad c_0 \approx 340 \text{ m/s}$$

Another analysis of the speed of sound.

Density dependence of speed of sound for ideal gas :

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0\gamma}{\rho_0}\frac{p/p_0}{\rho/\rho_0} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

Some details of the analysis reveal that beyond the linear approximation, the velocity of sound is highly non-linear.