# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF Online or (occasionally) in Olin 103

# Discussion notes for Lecture 2
# Two particle interactions and scattering theory

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday
Tim – 9 AM Tuesday
Zhi – 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday
Bamidele –

# Your questions

## From Nick –

1. From assignment one: Can you explain the motivation of the function we plotted?
2. If particle 1 dominates the motion in our scattering problems, are we assuming (for now at least) that particle 2 has no velocity?
3. What does closest approach mean?
4. What exactly is the scattering parameter, b?
5. What's blue and what's green in the figure on slide 8?
6. For assignment 2: Is E=1/2m1v1^2 + V(r) and where does b come into play?

## From Tim –

1. I am unclear as to why the quantity vanishes when the force F*ji* lies along r*i*-r*j* as in the equation(1.2). (from page 6)

# Your questions – continued

## From Derek

1. Slide 17 mentions that for quantum mechanical hard spheres at low energy the cross sectional area is 4 times as large and I was wondering why that is the case and what happens to the cross sectional area at higher energies.

## From Bamidele

1. Slide 6, when we say M2>>M1, I expect that M2 dominates the motion not M1 (ie m1 is negligible). How's this? Or am I thinking of it in the wrong way?

2. What's the significance of using Hard spheres as an example? Are they descriptive of most scattering experiments?

# PHY 711 Classical Mechanics and Mathematical Methods

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| **MWF 10 AM-10:50 AM** | **OPL 103** | **http://www.wfu.edu/~natalie/f20phy711/** |

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| **Instructor:** Natalie Holzwarth | **Office:**300 OPL | **e-mail:**natalie@wfu.edu |

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## Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

| | Date | F&W Reading | Topic | Assignment | Due |
|---|---|---|---|---|---|
| **1** | Wed, 8/26/2020 | Chap. 1 | Introduction | #1 | 8/31/2020 |
| **2** | Fri, 8/28/2020 | Chap. 1 | Scattering theory | #2 | 9/02/2020 |
| **3** | Mon, 8/31/2020 | Chap. 1 | Scattering theory | | |
| **4** | Wed, 9/02/2020 | Chap. 1 | Scattering theory | | |
| **5** | Fri, 9/04/2020 | Chap. 2 | Non-inertial coordinate systems | | |
| **6** | Mon, 9/07/2020 | Chap. 3 | Calculus of Variation | | |

# PHY 711 – Assignment #2

08/28/2020

1. Consider a particle of mass $m$ moving in the vicinity of another particle of mass $M$ where $m << M$. The particles interact with a conservative central potential of the form

$$V(r) = V_0\left(\left(\frac{r_0}{r}\right)^2 - \left(\frac{r_0}{r}\right)\right),$$

where $r$ denotes the magnitude of the particle separation and $V_0$ and $r_0$ denote energy and length constants, respectively. The total energy of the system is $V_0$.

   (a) First consider the case where the impact parameter $b = 0$. Find the distance of closest approach of the particles.

   (b) Now consider the case where the impact parameter $b = r_0$. Find the distance of closest approach of the particles.

From your questions -- From assignment one: Can you explain the motivation of the function we plotted?

# PHY 711 – Assignment #1

1. Use maple or mathematica to evaluate and plot the integral

$$g(x) = \int_0^{\pi} \cos(x\cos(t))dt.$$

Note that the result is a "special function".

Comment -- My thought was to motivate the use of powerful modern tools of algebraic manipulation software which encapsulates centuries of detailed work by brilliant mathematicians. In this case, the answer is a Bessel function which will come up in this class and perhaps your other work. For more information about special functions see the website – https://dlmf.nist.gov/

# Comment on quiz questions

1.
$$g(t)=\int_0^t \left(x^2+t\right)dx \qquad \frac{dg}{dt}=\int_0^t \frac{d\left(x^2+t\right)}{dt}dt+\left(x^2+t\right)\Big|_{x=t}$$

$$=\int_0^t dt+\left(t^2+t\right)=t^2+2t$$

2. Evaluate the integral $\oint \frac{dz}{z}$ for a closed contour about the origin.

Suppose that $z=e^{i\theta}$ $\qquad dz=e^{i\theta}id\theta \qquad \oint \frac{dz}{z}=\int_0^{2\pi}\frac{e^{i\theta}id\theta}{e^{i\theta}}=2\pi i$

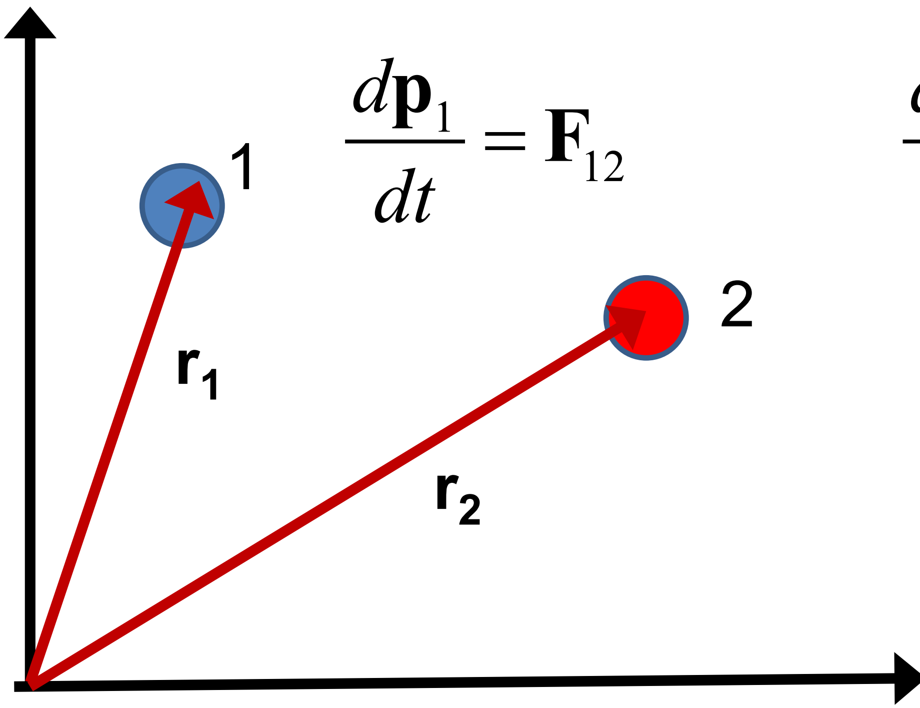
3. $\frac{df}{dx}=f \qquad \Rightarrow f(x)=Ae^x \qquad f(x=0)=A=1 \quad \Rightarrow A=1$

4. $\sum_{n=1}^{N} a^n=\frac{a-a^{N+1}}{1-a}$ $\qquad$ Let $S\equiv\sum_{n=1}^{N}a^n$ $\quad$ Note that $\quad aS-S=a^{N+1}-a$

First consider fundamental picture of particle interactions

Classical mechanics of a conservative 2-particle system.

$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$




$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \qquad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

In general, we will assume that $V(\mathbf{r})=V(r)$ (a central potential).

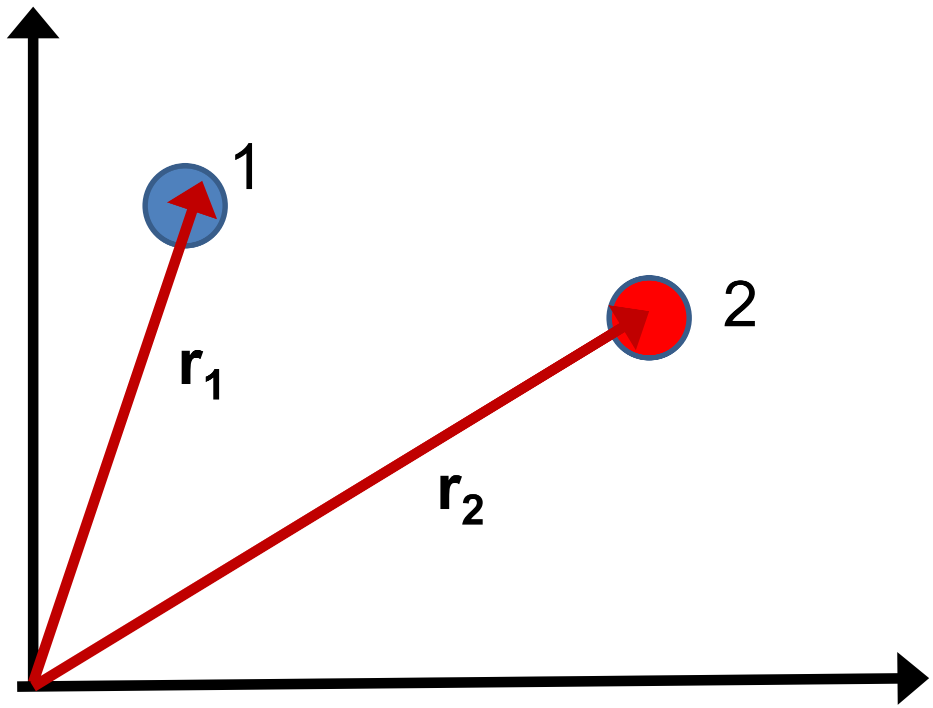
Your questions -- I am unclear as to why the quantity vanishes when the force F*ji* lies along r*i*-r*j* as in the equation(1.2). (from page 6)

Comments – I believe that this is a reference to the textbook's footnote on page 6 which makes an important point that we have so far glossed over. In the previous slide we assumed a central potential – *V(r)* which depends only on the distance of the two particles and has no directional dependence. This often happens in nature and when it does, angular momentum

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i$$

is a constant of the motion because the interaction force on the system applies no torque. We assume this is the case for our derivations in this chapter.

Energy is conserved: $$E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2+V\left(\mathbf{r}_1-\mathbf{r}_2\right)$$




For a central potential $V(\mathbf{r})=V(r)$, angular momentum is conserved. For the moment we also make the simplifying assumption that $m_2 >> m_1$ so that particle 1 dominates the motion.

Your questions -- If particle 1 dominates the motion in our scattering problems, are we assuming (for now at least) that particle 2 has no velocity?

Slide 6, when we say M2>>M1, I expect that M2 dominates the motion not M1 (ie m1 is negligible). How's this? Or am I thinking of it in the wrong way?

Comment -- Again, apologies for unwarranted leaps on my part. In order to keep things simple for now, we are thinking that particle 2 is stationary as the “target” particle while particle 1 is doing the scattering. In the reference frame that particle 2 is initially stationary, it is a good approximation to assume that particle 2 remains stationary after interacting with particle 1 if particle 2 is very massive because the total momentum must be conserved.

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \leq a \\ 0 & r > a \end{cases}$

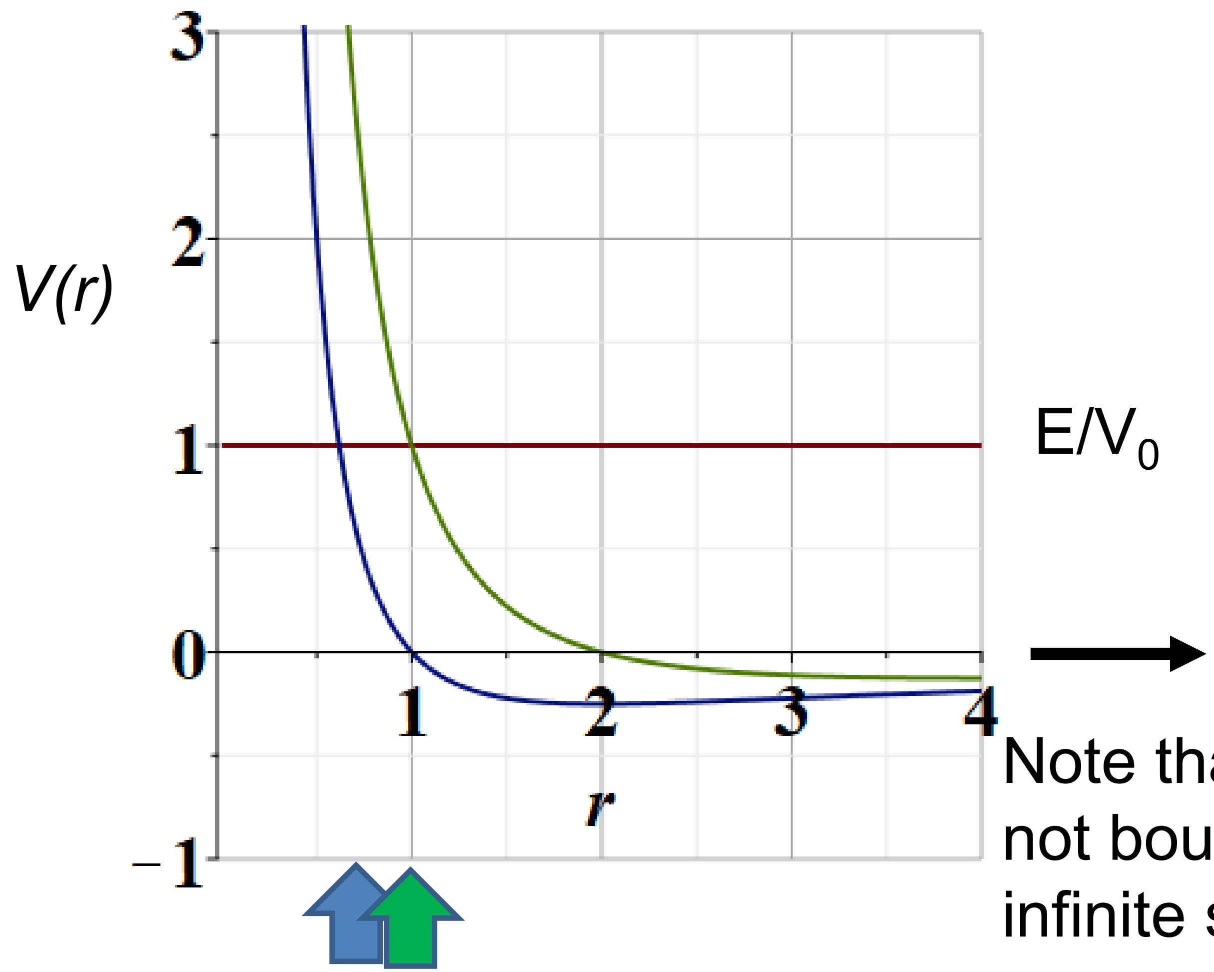
Coulomb or gravitational: $V(r) = \frac{K}{r}$

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

Your question -- What's the significance of using Hard spheres as an example? Are they descriptive of most scattering experiments?

Comment – If we were face to face, I would have my box of marbles. Another example of hard sphere motion would be billiard balls or hockey pucks. In reality, internal spin also needs to be taken into account. The hard sphere model is mathematically convenient, and can be an approximation to some particle interactions.

Representative plot of *V(r)*




Note that particles are not bound; can reach infinite separation

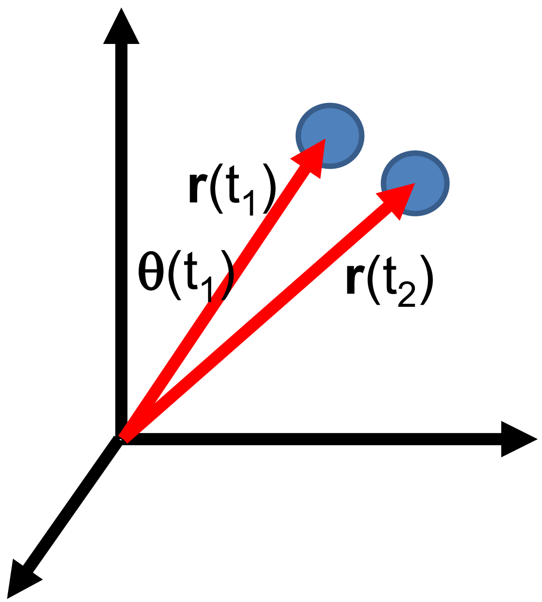
Distances of closest approach

Your question -- What's blue and what's green in the figure on slide 8?

Comment – Again, apologies for unwarranted leaps. Here some more complete comments.

Here we are assuming that the target particle is stationary and $m_1 \equiv m$.

The origin of our coordinate system is taken at the position of the target particle.




Conservation of energy:

$$E = \frac{1}{2} m \left( \frac{d\mathbf{r}}{dt} \right)^2 + V(r)$$

$$= \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right) + V(r)$$

Conservation of angular momentum:

$$L = mr^2 \frac{d\theta}{dt}$$

# Comments continued --

Conservation of energy:

$$E = \frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V(r)$$

$$= \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) + V(r)$$

$$= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \boxed{\frac{L^2}{2mr^2} + V(r)} \quad \Leftarrow \quad V_{eff}(r)$$

Conservation of angular momentum:

$$L = mr^2\frac{d\theta}{dt}$$

Also note that when $r \to \infty, \quad V(r) \to 0$

$$\mathbf{L} \equiv \mathbf{r} \times m\frac{d\mathbf{r}}{dt} \qquad L = b\sqrt{2mE}$$

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{b^2E}{r^2} + V(r)$$

Which of the following are true:

a. The particle moves in a plane.
b. For any interparticle potential the trajectory can be determined/calculated.
c. Only for a few special interparticle potential forms can the trajectory be determined.

Why should we care about this?

a. We shouldn’t really care.
b. It is only of academic interest
c. It is of academic interest but can be measured.
d. Many experiments can be analyzed in terms of the particle trajectory.

# Scattering theory:

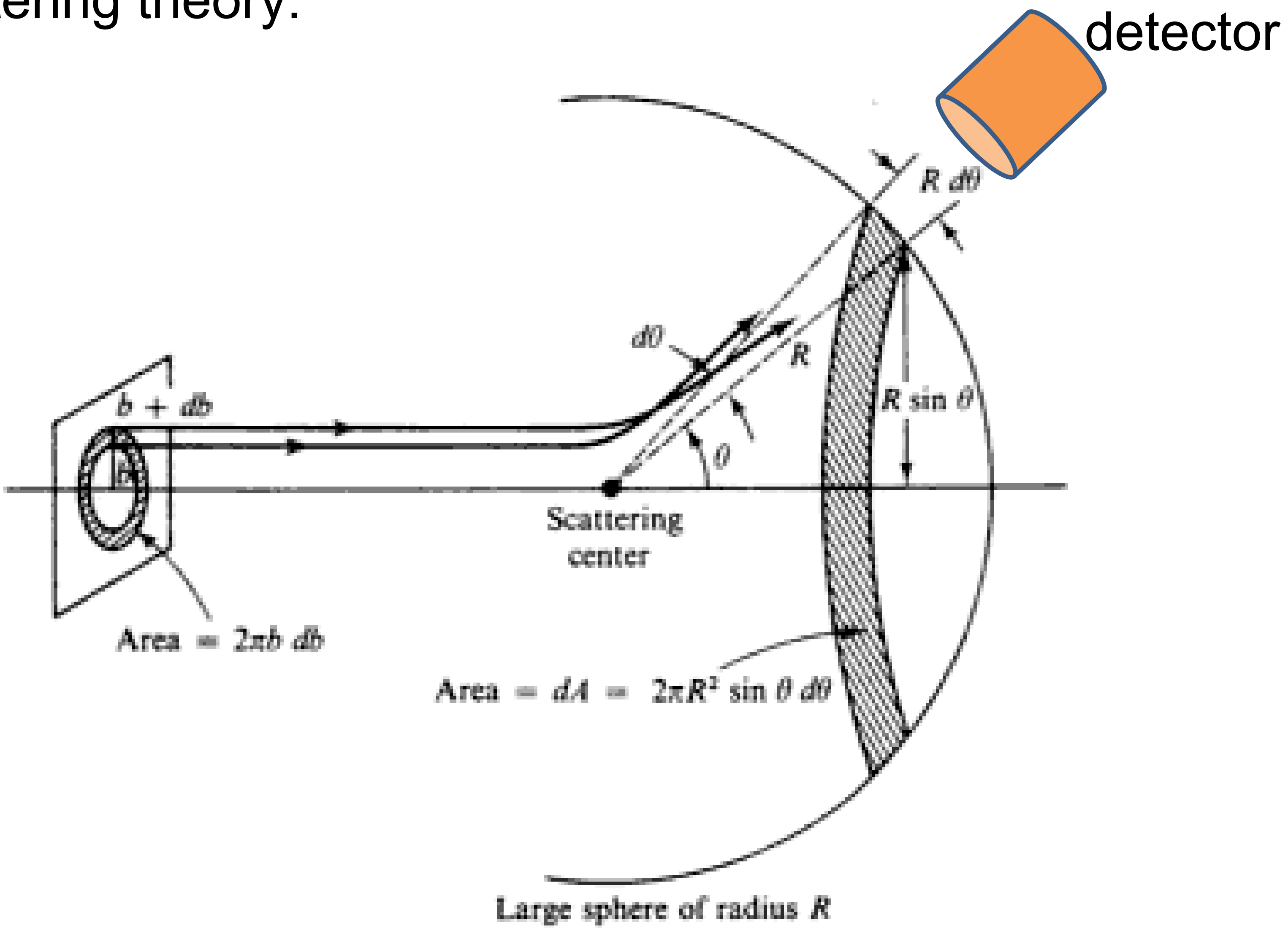
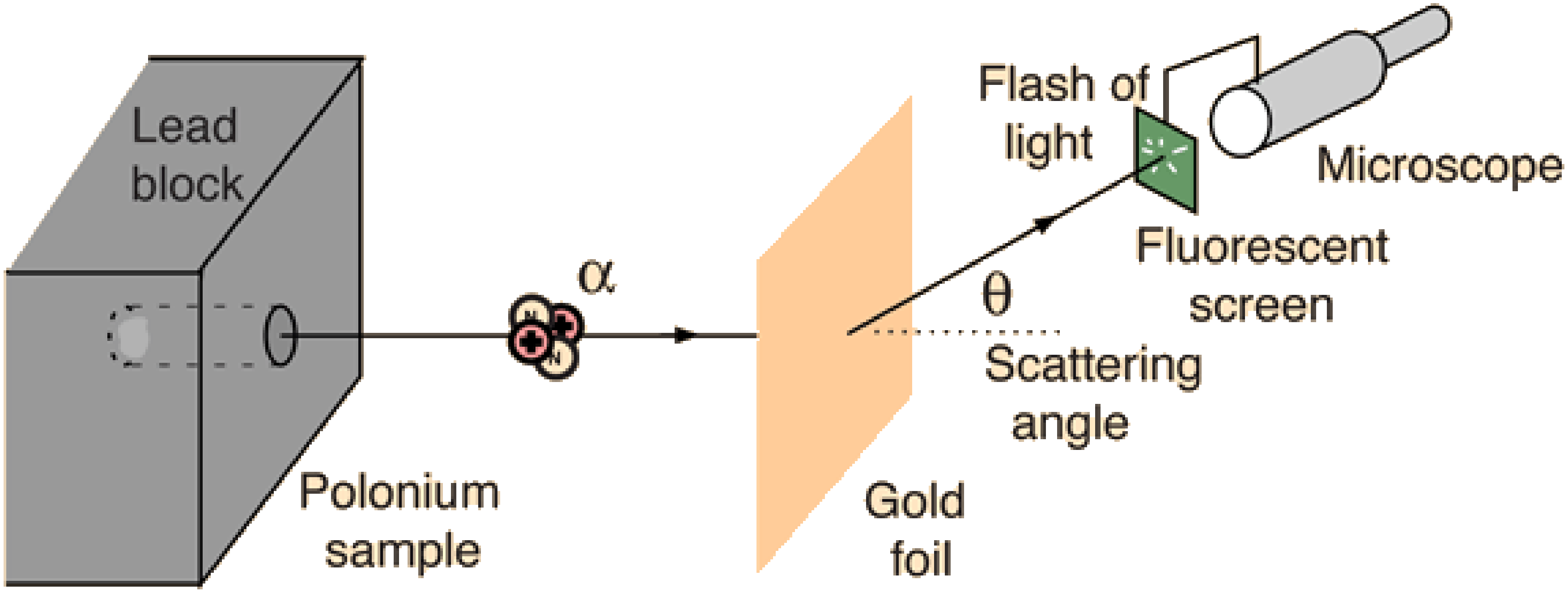
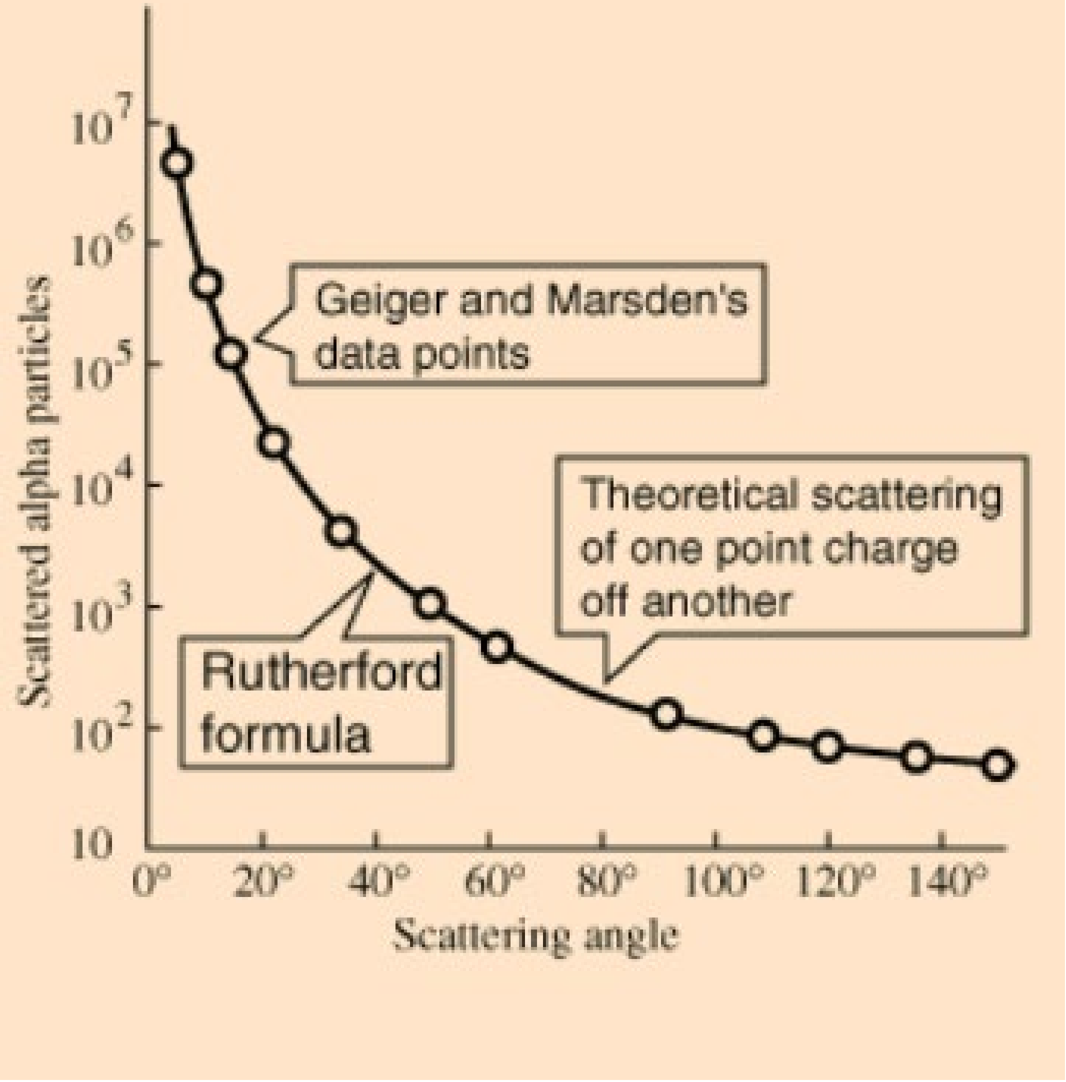



**Figure 5.5** The scattering problem and relation of cross section to impact parameter.

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



# Graph of data from scattering experiment




**From website: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/rutsca2.html**

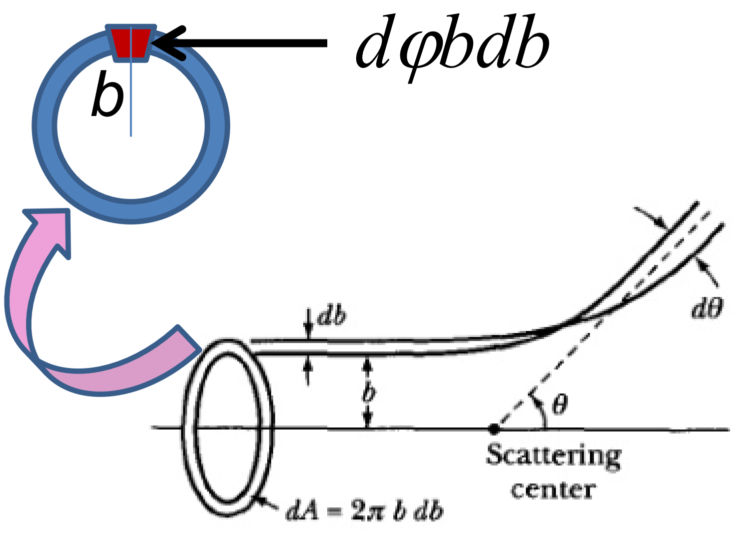
# Standardization of scattering experiments --

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$

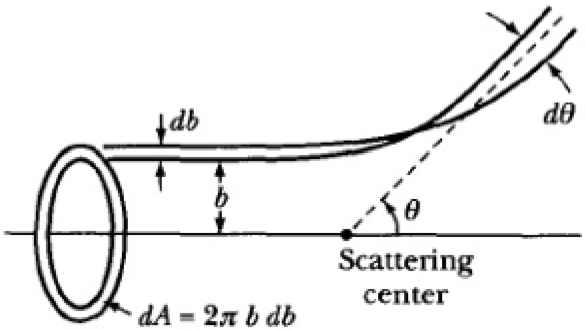
Impact parameter: *b*

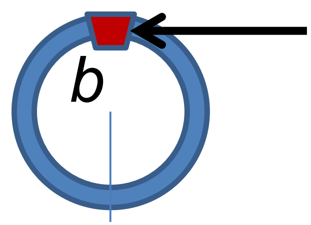



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

**Note:** The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



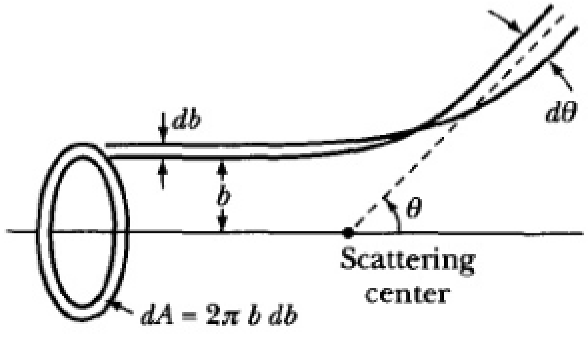

$d\varphi b db$

Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in $\varphi$

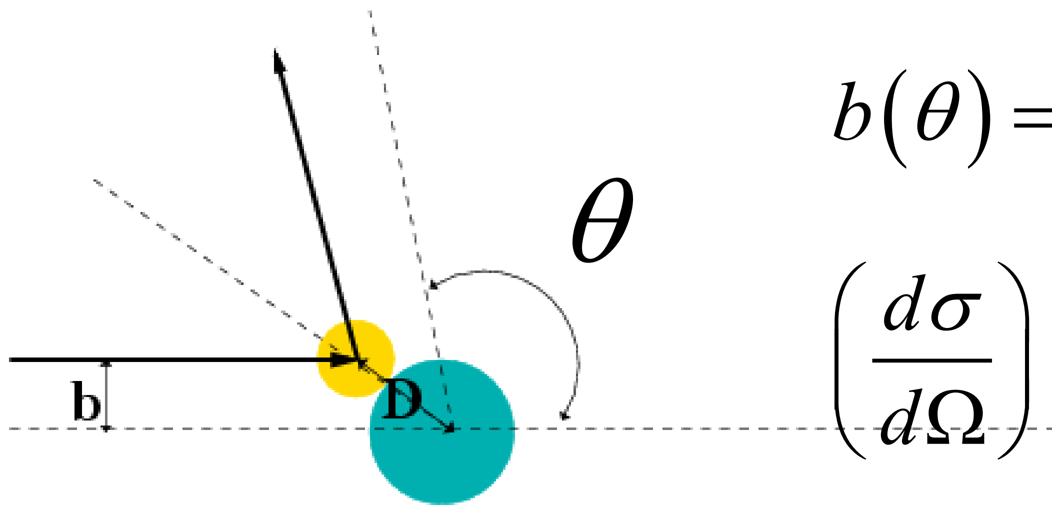
Simple example – collision of hard spheres having mutual radius D; very large target mass




$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

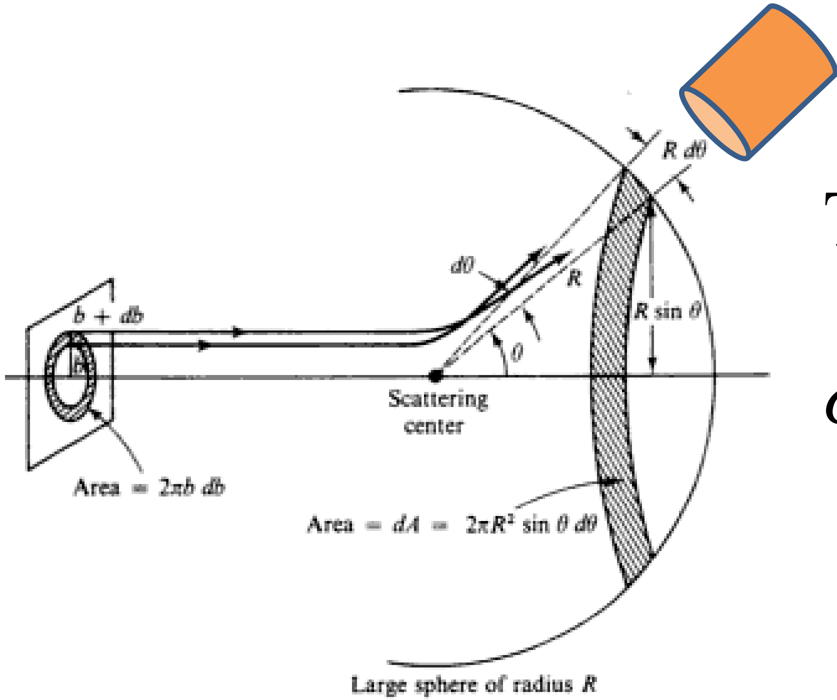
$$b(\theta) = ?$$




$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

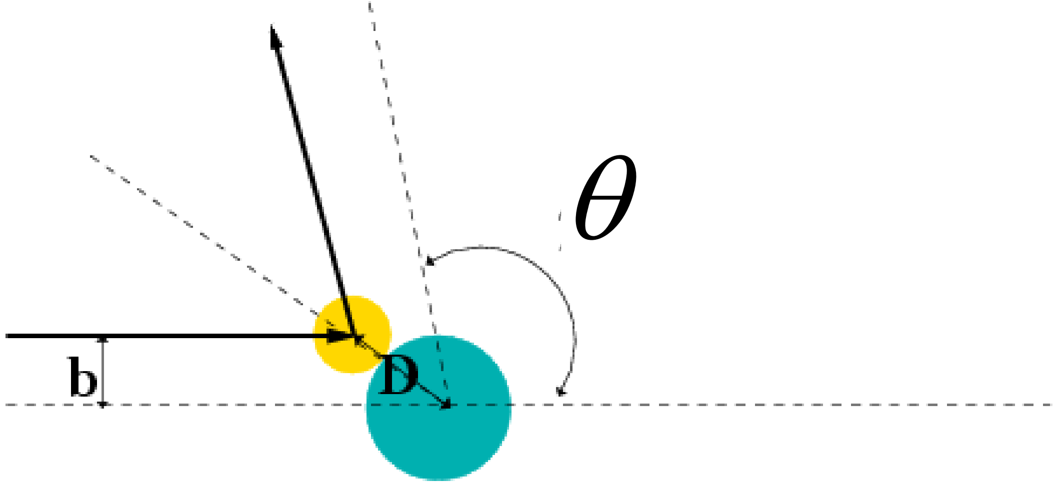
# Simple example – collision of hard spheres -- continued




Total scattering cross section:

$$\sigma = \int \left( \frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:




$$\left( \frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

## More details of hard sphere scattering –

Hidden in the analysis are assumptions about the scattering process such as:

- No external forces ➔ linear momentum is conserved
- No dissipative phenomena ➔ energy is conserved
- No torque on the system ➔ angular momentum is conserved
- Target particle is much more massive than scattering particle
- Other assumptions??

Note that for quantum mechanical hard spheres at low energy the total cross section is 4 times as large.

Your question -- Slide 17 mentions that for quantum mechanical hard spheres at low energy the cross sectional area is 4 times as large and I was wondering why that is the case and what happens to the cross sectional area at higher energies.

Comment – A qualitative answer is that in quantum mechanics at low energy there are interference effects that cause the cross section to be larger. At higher energies, classical and quantum mechanics become more similar.