PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Discussion for Lecture 30 -- Chap. 9 of F&W

Wave equation for sound in linear approximation

1. Wave equations for sound
2. Standing wave solutions
3. Traveling wave solutions

Tentative schedule --

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves		
32	Fri, 11/06/2020	Chap. 9	Non linear effects in sound waves		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks		
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids		
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

No HW; work on "mini-projects".

PHY 711 -- Assignment #20

Nov. 02, 2020

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider a cylindrical pipe of length 0.5 m and radius 0.05 m, open at both ends. For air at 300 K and atmospheric pressure in this pipe, find several of the lowest frequency resonances.

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (EST)
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Tim – 11 AM Wednesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Your questions –
From Gao –

1. The open-open pipe boundary conditions [were given in the lecture]. But what physical mechanism leads to this kind of condition?
2. Can wave velocity in an ideal gas only be the speed of sound, or can be other speeds?

Review –

Hydrodynamic equations for isentropic air + linearization about equilibrium → wave equation for air (sound waves)

Which of the following things correctly describe the wave equation for sound in air and the wave equation for elastic media?

a. The wave velocity is different for sound in air and waves in elastic media.
b. The wave motion in elastic media can be either transverse or longitudinal.
c. The wave motion for sound in air can be either transverse or longitudinal.

Wave equation for air :

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$$\mathbf{v} = -\nabla\Phi$$

Note that, we also have :

$$\frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2\delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}}\cdot\ \mathbf{V} = \ \hat{\mathbf{n}}\cdot\delta\mathbf{v} = -\ \hat{\mathbf{n}}\cdot\nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0\frac{\partial\Phi}{\partial t} = 0$$

Your question --Can wave velocity in an ideal gas only be the speed of sound, or can [there] be other speeds?

Comment -- The term "speed of sound" refers to the linear approximation to the hydrodynamic equations for an ideal gas. We also assumed isoentropic conditions. In this case, only longitudinal waves are possible. For other types of media other "sound" speeds are possible, particularly, longitudinal and transverse waves often have different speeds.

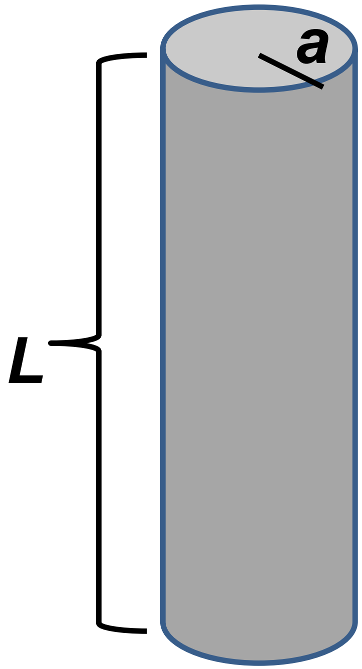
Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Time harmonic standing waves in a pipe



$$\boxed{\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0}$$

Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \text{Define:} \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates:

$$\Phi(r,\phi,z,t) = R(r)F(\phi)Z(z)e^{-i\omega t} \equiv R(r)F(\phi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\phi,z,t) = 0$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi(r,\varphi,z,t)=0$$

$$\Phi(r,\varphi,z,t)=R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi)=e^{im\varphi};\ \ F(\varphi)=F(\varphi+2\pi N)\Rightarrow m=\text{integer}$$

$$Z(z)=e^{i\alpha z};\ \ \alpha=\text{real plus other restrictions}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

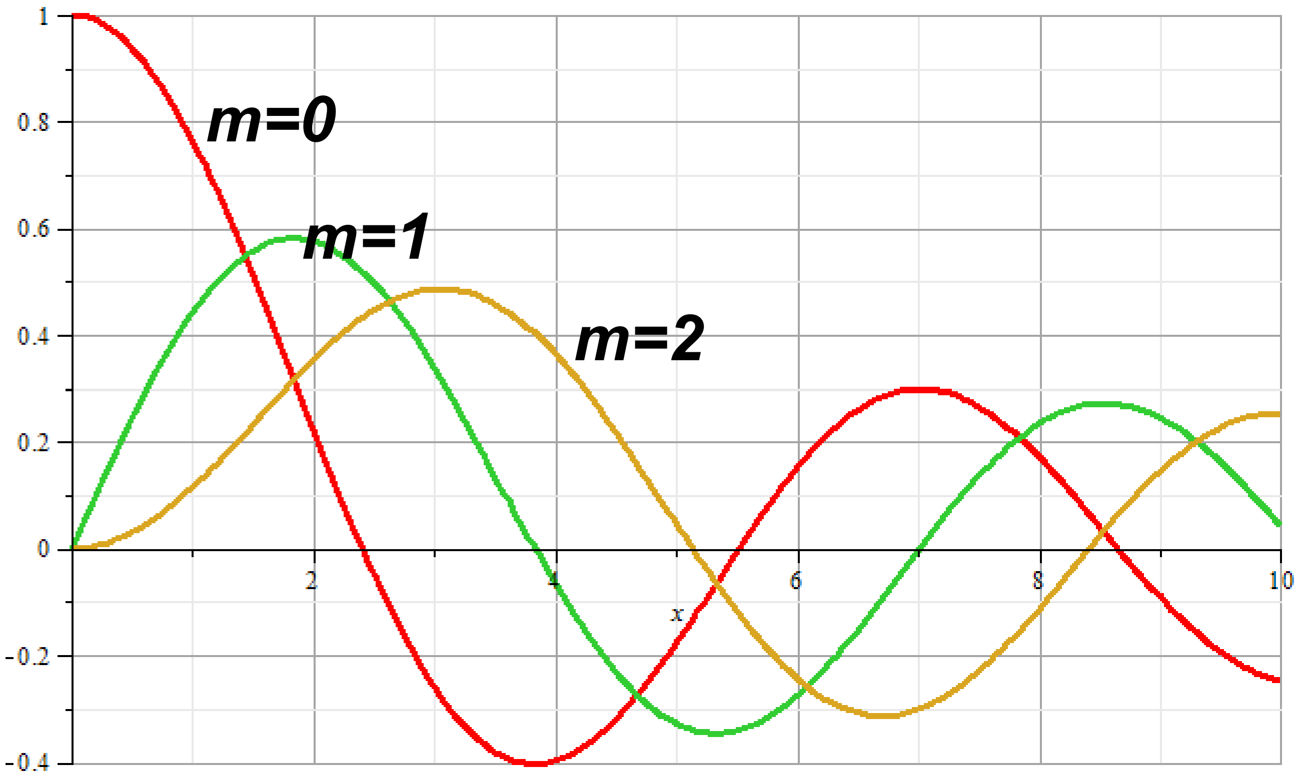
For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2-\alpha^2$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

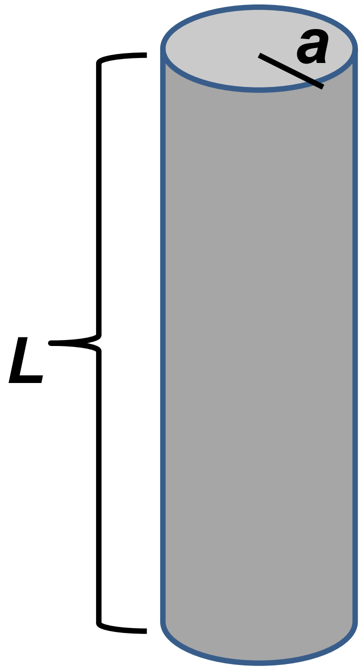
Cylinder surface boundary conditions: $\left.\frac{dR}{dr}\right|_{r=a}=0$

$\Rightarrow R(r)=J_m(\kappa r)$ where for $\frac{dJ_m(x'_{mn})}{dx}=0,\quad \kappa_{mn}=\frac{x'_{mn}}{a}$

Bessel functions : $J_m(x)$



Now recall the boundary conditions



Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

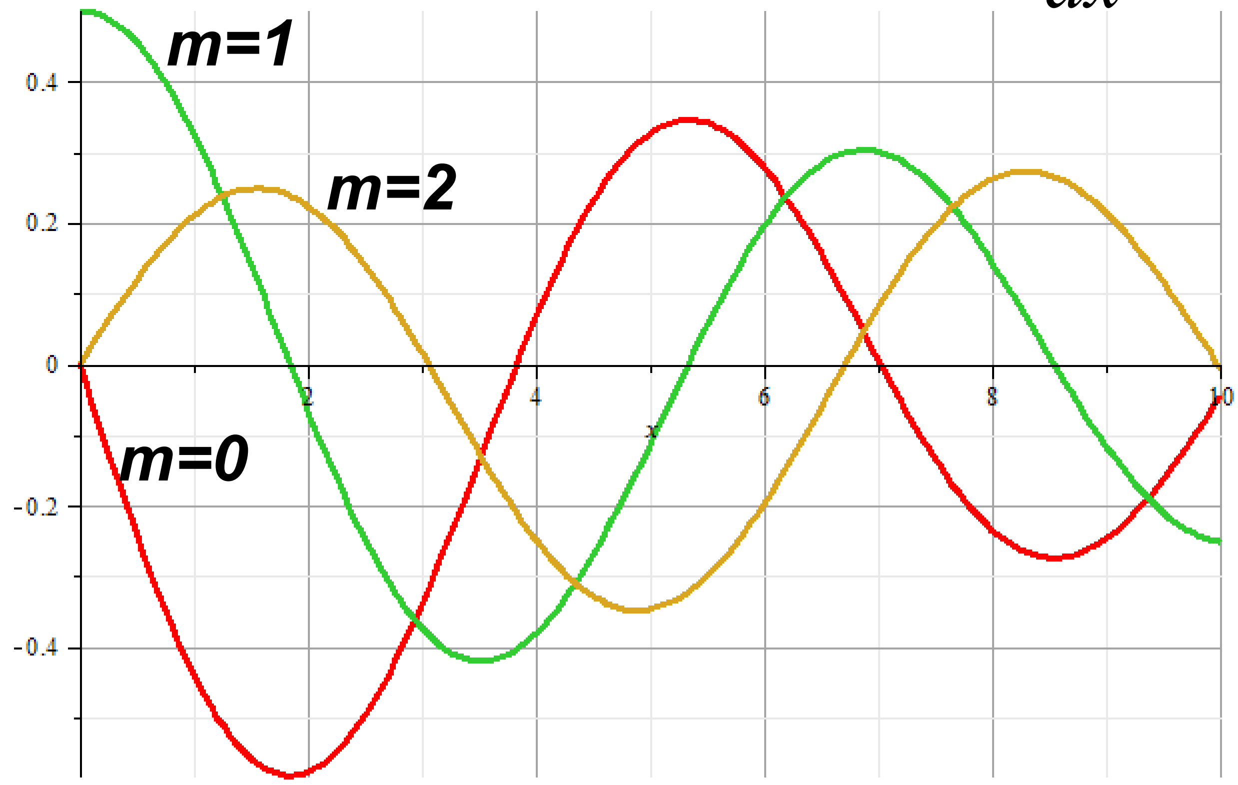
At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$= J_m(\kappa r)e^{im\varphi}e^{i\alpha z}e^{-i\omega t}$$

$$\text{For} \quad r = a, \quad \left.\frac{\partial \Phi(r,\varphi,z,t)}{\partial r}\right|_{r=a} = 0$$

Bessel function derviatives : $\dfrac{dJ_m(x)}{dx}$



Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Boundary condition for z=0, z=L:

For open-open pipe:

$$Z(0) = Z(L) = 0 \qquad \Rightarrow \; Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \qquad p = 1,2,3...$$

Resonant frequencies:

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05$$

Alternate boundary condition for z=0, z=L:

For open - closed pipe:

$$\frac{dZ(0)}{dz} = Z(L) = 0 \qquad \Rightarrow \; Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \qquad p = 0,1,2,3\ldots$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

The above analysis pertains to resonant air waves within a cylindrical pipe. As previously mentioned, you can hear these resonances if you put your ear close to such a pipe. The same phenomenon is the basis of several musical instruments such as organ pipes, recorders, flutes, clarinets, oboes, etc.

Question – what about a trumpet, trombone, French horn, etc?

a. Same idea?
b. Totally different?

But for musical instruments, you do not want to put your ear next to the device – additional considerations must apply. Basically, you want to couple these standing waves to produce traveling waves.

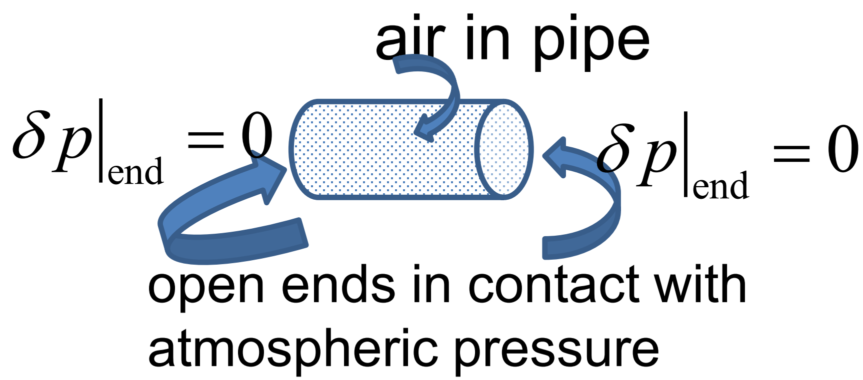
Modifications needed for the pandemic --

Image from the Winston-Salem Journal 11/1/2020

Your questions –
From Gao –

1. The open-open pipe boundary conditions [were given in the lecture]. But what physical mechanism leads to this kind of condition?
2. Can wave velocity in an ideal gas only be the speed of sound, or can be other speeds?

Comment --
1. Open pipe boundary condition



More relationships

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho$ Here $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

In terms of the velocity potential:

$$\delta \mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\Rightarrow \delta p = \rho_0 \frac{\partial \Phi}{\partial t}$$

Simple model of a sound amplifier --

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$

can be represented as boundary value of $\Phi(\mathbf{r},t)$