PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 30 -- Chap. 9 of F&W

Wave equation for sound in linear approximation

1. **Wave equations for sound**
2. **Standing wave solutions**
3. **Traveling wave solutions**

In this lecture, we will consider some solutions to the linear sound wave equations.

Tentative schedule --

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves		
32	Fri, 11/06/2020	Chap. 9	Non linear effects in sound waves		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks		
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids		
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Here is a tentative schedule for the next several weeks.

PHY 711 -- Assignment #20

Nov. 02, 2020

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider a cylindrical pipe of length 0.5 m and radius 0.05 m, open at both ends. For air at 300 K and atmospheric pressure in this pipe, find several of the lowest frequency resonances.

Homework problem based on today's lecture.

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Review of the equations we derived last time.

Solutions to wave equation:

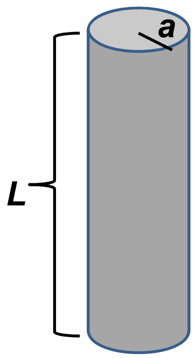
$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

In general, we will be interested in time harmonic solutions to the wave equation, where omega denotes the pure frequency of the wave.

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values :

At fixed surface : $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

At free surface : $\frac{\partial \Phi}{\partial t} = 0$

For example, consider a pipe of length L and radius a. In this pipe, we are interested in the behavior of the air. Should you have such a piper at home, put your ear close to one end. What do you hear?

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \text{Define :} \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates:

$$\Phi(r,\phi,z,t) = R(r)F(\phi)Z(z)e^{-i\omega t} \equiv R(r)F(\phi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\phi,z,t) = 0$$

Here we consider the equations of linear air within the paper. Cylindrical coordinates are the natural analysis tools for this case.

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi(r,\varphi,z,t)=0$$

$$\Phi(r,\varphi,z,t)=R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi)=e^{im\varphi};\; F(\varphi)=F(\varphi+2\pi N)\Rightarrow m=\text{integer}$$

$$Z(z)=e^{i\alpha z};\;\; \alpha=\text{real plus other restrictions}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

11/02/2020 PHY 711 Fall 2020 -- Lecture 30 8

The equation is separable in the radial, angular, z, and time variables. Because of the cylindrical geometry, the angular part takes the form of exp(I m phi), where m has to be an integer. We also are motivated to assume that the Z(z) function has a sinusoidal form with an unknown constant alpha. Finally, the equation for the radial equation now takes a familiar form.

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2-\alpha^2$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

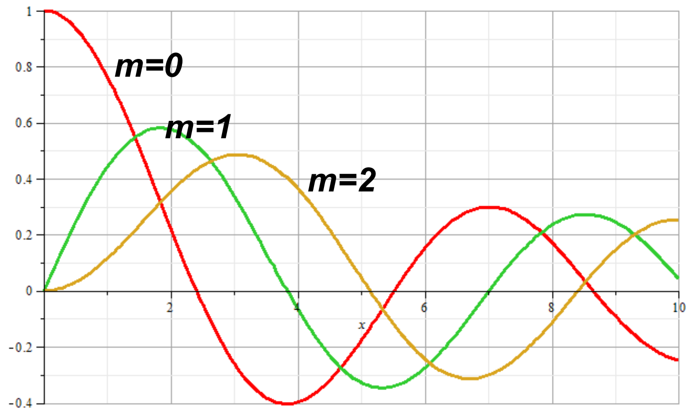
Cylinder surface boundary conditions: $\left.\frac{dR}{dr}\right|_{r=a}=0$

$$\Rightarrow R(r)=J_m(\kappa r) \quad \text{where for} \quad \frac{dJ_m(x'_{mn})}{dx}=0, \quad \kappa_{mn}=\frac{x'_{mn}}{a}$$

11/02/2020 PHY 711 Fall 2020 -- Lecture 30 9

With certain assumptions, we can show that the radial solutions for the air motion, are Bessel functions of order m.

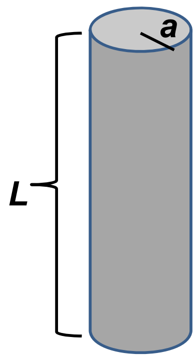
Bessel functions : $J_m(x)$



11/02/2020 PHY 711 Fall 2020 -- Lecture 30 10

Some plots of Bessel functions.

Now recall the boundary conditions



Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

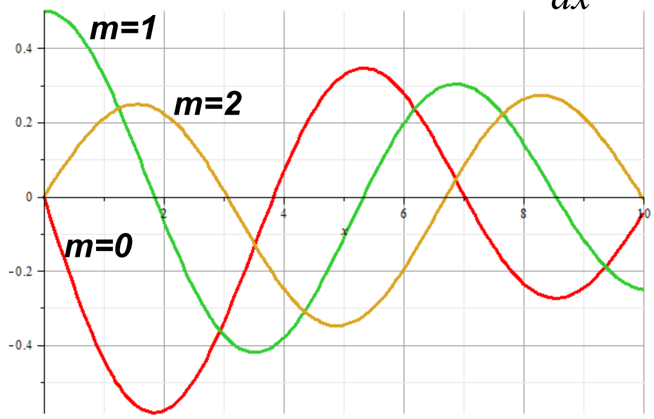
At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$
$$= J_m(\kappa r)e^{im\varphi}e^{i\alpha z}e^{-i\omega t}$$

For $r = a$, $\left.\dfrac{\partial \Phi(r,\varphi,z,t)}{\partial r}\right|_{r=a} = 0$

Now consider the boundary conditions for the sound wave within the pipe, focusing on the radial direction.

Bessel function derviatives : $\frac{dJ_m(x)}{dx}$



Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Zeroes of the derivatives of Bessel functions.

Boundary condition for z=0, z=L:

For open - open pipe :

$$Z(0) = Z(L) = 0 \qquad \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \qquad p = 1,2,3...$$

Resonant frequencies :

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

We also need to consider the boundary conditions for the air motion in the z direction where the paper can be either open or closed. For the open, open pipe, we then find the resonant wavevectors.

Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05$$

More details.

Alternate boundary condition for z=0, z=L:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \qquad p = 0,1,2,3...$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

Now consider other boundary conditions and their resonances.

The above analysis pertains to resonant air waves within a cylindrical pipe. As previously mentioned, you can hear these resonances if you put your ear close to such a pipe. The same phenomenon is the basis of several musical instruments such as organ pipes, recorders, flutes, clarinets, oboes, etc.

Question – what about a trumpet, trombone, French horn, etc?

a. Same idea?
b. Totally different?

But for musical instruments, you do not want to put your ear next to the device – additional considerations must apply. Basically, you want to couple these standing waves to produce traveling waves.

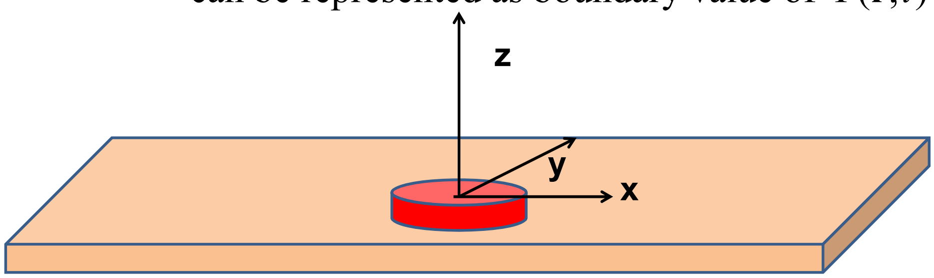
Simple model of a sound amplifier --

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$ can be represented as boundary value of $\Phi(\mathbf{r},t)$



11/02/2020 PHY 711 Fall 2020 -- Lecture 30 17

This is what we will consider on Wednesday.