

PHY 711 Classical Mechanics and Mathematical Methods

**10-10:50 AM MWF online or (occasionally)
in Olin 103**

Discussion for Lecture 31: Chap. 9 of F&W

**Wave equation for sound in the linear
approximation**

- 1. Sound generation**
- 2. Sound scattering**



Thursday, 11/4/2020
4 PM online

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“Status and Perspective of Organic Photovoltaic: Is it Ready for Commercialisation?”

The current success of organic semiconductor technology is mainly driven by the development of organic light emitting diodes (OLED), which are now routinely employed in display technologies. In the last decade, however, organic photovoltaics (OPV), leveraging the impressive improvement in device efficiency and stability, have gradually moved from a lab curiosity to a niche market.^[1] Their recent success has coincided with the rapid development of effective replacements for the fullerene-based materials that have been prevalent as electron acceptor materials until recently; namely the small molecule nonfullerene acceptors (NFAs).^[2] Through strategic design, an acceptor-donor-acceptor (A-D-A) configuration afforded highly absorbing small molecules with tunable energetics, thereby allowing the achievement of record power conversion efficiencies (PCEs) in OPVs. This

Your questions –


From Tim –

1. You describe the wave as longitudinal. So would the displacement vector \mathbf{r} that is used best be described in cylindrical coordinates?
2. It seems that the Laplacian being applied is in cylindrical coordinates. I was also curious about the Laplacian relation discussed on slide 14.

From Gao –

1. About today's lecture, where does this equation come from?

$$\nabla^2 \tilde{\Phi} + k^2 \tilde{\Phi} = -\tilde{f}(\mathbf{r}, \omega)$$

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
 31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Non linear effects in sound waves		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks		
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids		
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule for weekly one-on-one meetings (EST)

Tim – 9 AM Tuesday

Gao – 9 PM Tuesday

Nick – 11 AM Wednesday

Jeanette – 11 AM Friday

Derek – 12 PM Friday

Comment about the units of sound frequency

Harmonic time dependence of a wave:

$$\Phi(\mathbf{r}, t) = f(\mathbf{r})e^{-i\omega t} = f(\mathbf{r})e^{-2\pi i\nu t}$$

Note that ω has units of radians/sec

ν has units of cycles/sec (Hz)

$$\nu = \frac{\omega}{2\pi}$$

Solutions to wave equation:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r}, t) = A e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c} \right)^2$$

Note that these sound waves are "longitudinal"

-- the velocity wave direction is along the propagation direction:

$$\delta \mathbf{v} = -\nabla \Phi = -iA \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}$$

Some comments about Monday's lecture
 Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta \mathbf{v}}{\partial t} = - \frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

Note that:

In terms of the velocity potential:

$$\delta \mathbf{v} = -\nabla \Phi$$

$$\delta \mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla \tilde{\Phi}(\mathbf{r}, t)$$

$$\text{for } \tilde{\Phi}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \int^t dt' K(t')$$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = - \frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \quad \nabla \left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0$$

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0$$

$$-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} = K(t)$$

$$-\frac{\partial \tilde{\Phi}}{\partial t} + \frac{\delta p}{\rho_0} = 0$$

Some comments about Monday's lecture -- continued

Expressing pressure in terms of the density:

$$p = p(s, \rho) = p_0 + \delta p \quad \text{where } s \text{ denotes the (constant) entropy}$$

$$p_0 = p(s, \rho_0)$$

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_s \delta \rho \equiv c^2 \delta \rho$$

$$\nabla \left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0 \quad \Rightarrow \quad \nabla \left(-\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta \rho}{\rho_0} \right) = 0$$

$$\left(-\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta \rho}{\rho_0} \right) = K(t) \quad \Rightarrow \quad -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0} \frac{\partial \delta \rho}{\partial t} = 0$$

$$\frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Some comments about Monday's lecture -- continued

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Additional relations:

$$\delta p = c^2 \delta \rho = \rho_0 \frac{\partial \Phi}{\partial t}$$

$$\Rightarrow \frac{\partial^2 \delta \rho}{\partial t^2} - c^2 \nabla^2 \delta \rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity \mathbf{V} :

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \quad \Rightarrow \quad \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r}, t) = \int d^3 r' \int dt' G(\mathbf{r} - \mathbf{r}', t - t') f(\mathbf{r}', t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Your question – Where does this force term come from?

Comment --

Equations to lowest order in perturbation -- keeping applied force:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} & \Rightarrow & \quad \frac{\partial \delta \mathbf{v}}{\partial t} = \mathbf{f}_{\text{applied}} - \frac{\nabla \delta p}{\rho_0} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 & \Rightarrow & \quad \frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0 \end{aligned}$$

Assuming $\delta \mathbf{v} = -\nabla \Phi$ and $\delta p = \left(\frac{\partial p}{\partial \rho} \right)_s \delta \rho \equiv c^2 \delta \rho$ and $\mathbf{f}_{\text{applied}} = -\nabla U_{\text{applied}}$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = \mathbf{f}_{\text{applied}} - \frac{\nabla \delta p}{\rho_0} \Rightarrow -\nabla \left(\frac{\partial \Phi}{\partial t} - \frac{c^2 \delta \rho}{\rho_0} - U_{\text{applied}} \right) = 0$$

When the dust clears -- $\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = \frac{\partial U_{\text{applied}}}{\partial t}$

In fact, in our example the forcing term will be instead described in terms of a boundary value.

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Recall that

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega$$

Derivation of Green's function for wave equation -- continued

Define: $\tilde{G}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} G(\mathbf{r}, t) e^{i\omega t} dt$

$$G(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r}, \omega)$ must satisfy:

$$(\nabla^2 + k^2) \tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}') \quad \text{where} \quad k^2 = \frac{\omega^2}{c^2}$$

Derivation of Green's function for wave equation -- continued

$$(\nabla^2 + k^2)\tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r} - \mathbf{r}'$:

$$\tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = \frac{e^{\pm ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r} - \mathbf{r}'|$ and for $R > 0$:

$$(\nabla^2 + k^2)\tilde{G}(R, \omega) = \frac{1}{R} \frac{d^2}{dR^2} (R\tilde{G}(R, \omega)) + k^2\tilde{G}(R, \omega) = 0$$

Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$(\nabla^2 + k^2)\tilde{G}(R, \omega) = \frac{1}{R} \frac{d^2}{dR^2} (R\tilde{G}(R, \omega)) + k^2 \tilde{G}(R, \omega) = 0$$

$$\frac{d^2}{dR^2} (R\tilde{G}(R, \omega)) + k^2 (R\tilde{G}(R, \omega)) = 0$$

$$(R\tilde{G}(R, \omega)) = A e^{ikR} + B e^{-ikR}$$

$$\Rightarrow \tilde{G}(R, \omega) = A \frac{e^{ikR}}{R} + B \frac{e^{-ikR}}{R}$$

Derivation of Green's function for wave equation – continued
need to find A and B .

$$\text{Note that : } \nabla^2 \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = -\delta(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R, \omega) = \frac{e^{\pm ikR}}{4\pi R}$$

Your question – where did this come from?

Note that:
$$\nabla^2 \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} = -\delta(\mathbf{r} - \mathbf{r}')$$

Comment – the “proof” will be covered more carefully in PHY 712. The basic approach is to assume it is true and test that an integral on both sides of the equation in a small volume including $\mathbf{r}=\mathbf{r}'$ gives a consistent result.

Derivation of Green's function for wave equation – continued

$$\begin{aligned} G(\mathbf{r} - \mathbf{r}', t - t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r} - \mathbf{r}', \omega) e^{-i\omega(t-t')} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} e^{-i\omega(t-t')} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} e^{-i\omega(t-t')} d\omega \end{aligned}$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm i \frac{\omega}{c} |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Noting that $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega u} d\omega = \delta(u)$

$$\Rightarrow G(\mathbf{r} - \mathbf{r}', t - t') = \frac{\delta\left(t - \left(t' \mp \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

➔ In order to solve an inhomogeneous wave equation with a time harmonic forcing or boundary term, we can use the corresponding Green's function:

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{\pm ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

In fact, this Green's function is appropriate for solving equations with boundary conditions at infinity. For solving problems with surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V (h \nabla^2 g - g \nabla^2 h) d^3 r = \oint_S (h \nabla g - g \nabla h) \cdot \hat{\mathbf{n}} d^2 r$

$$\nabla^2 \tilde{\Phi} + k^2 \tilde{\Phi} = -\tilde{f}(\mathbf{r}, \omega)$$

$$(\nabla^2 + k^2) \tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \quad g \leftrightarrow \tilde{G}$$

$$\int_V (\tilde{\Phi}(\mathbf{r}, \omega) \delta(\mathbf{r} - \mathbf{r}') - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) f(\mathbf{r}, \omega)) d^3 r =$$

$$\oint_S (\tilde{\Phi}(\mathbf{r}, \omega) \nabla \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla \tilde{\Phi}(\mathbf{r}, \omega)) \cdot \hat{\mathbf{n}} d^2 r$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r}, \omega) \delta(\mathbf{r} - \mathbf{r}') - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) f(\mathbf{r}, \omega) \right) d^3 r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}, \omega) \nabla \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla \tilde{\Phi}(\mathbf{r}, \omega) \right) \cdot \hat{\mathbf{n}} d^2 r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}', \omega) \delta(\mathbf{r} - \mathbf{r}') - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) f(\mathbf{r}', \omega) \right) d^3 r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}', \omega) \nabla \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla \tilde{\Phi}(\mathbf{r}', \omega) \right) \cdot \hat{\mathbf{n}} d^2 r'$$

If the integration volume V includes the point $\mathbf{r} = \mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) f(\mathbf{r}', \omega) d^3 r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}', \omega) \nabla \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla \tilde{\Phi}(\mathbf{r}', \omega) \right) \cdot \hat{\mathbf{n}} d^2 r'$$

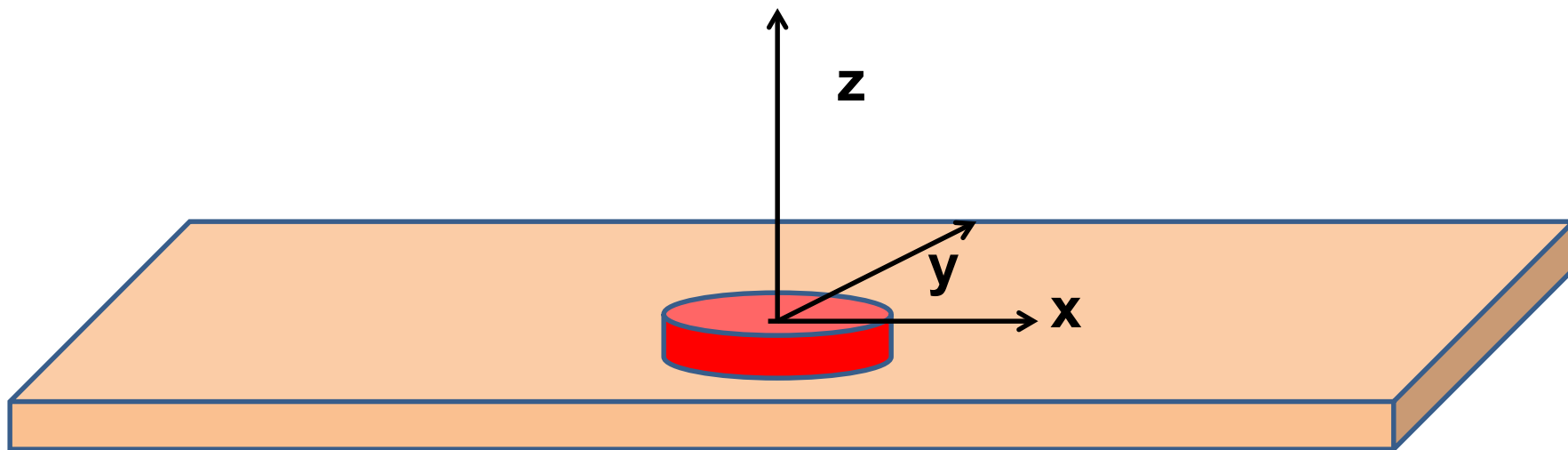
→ extra contributions from boundary

Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Example:

$f(\mathbf{r}, t) \Rightarrow$ time harmonic piston of radius a , amplitude $\varepsilon \hat{\mathbf{z}}$
can be represented as boundary value of $\Phi(\mathbf{r}, t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \tilde{f}(\mathbf{r}', \omega) d^3 r' + \oint_S \left(\tilde{\Phi}(\mathbf{r}', \omega) \nabla' \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla' \tilde{\Phi}(\mathbf{r}', \omega) \right) \cdot \hat{\mathbf{n}} d^2 r'$$

Boundary values for our example:

$$\left(\frac{\partial \tilde{\Phi}}{\partial z} \right)_{z=0} = \begin{cases} 0 & \text{for } x^2 + y^2 > a^2 \\ i\omega \epsilon a & \text{for } x^2 + y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = - \oint_{S: z'=0} \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \frac{\partial \tilde{\Phi}(\mathbf{r}', \omega)}{\partial z} dx' dy'$$

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega)_{z'=0} = \left. \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{2\pi|\mathbf{r} - \mathbf{r}'|} \right|_{z'=0}; \quad z > 0$$

$$\begin{aligned}\tilde{\Phi}(\mathbf{r}, \omega) &= - \oint_{S: z'=0} \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \frac{\partial \tilde{\Phi}(\mathbf{r}', \omega)}{\partial z} dx' dy' \\ &= -i\omega\epsilon a \int_0^a r' dr' \int_0^{2\pi} d\phi' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{2\pi|\mathbf{r} - \mathbf{r}'|} \Big|_{z'=0}\end{aligned}$$

Integration domain: $x' = r' \cos \varphi'$
 $y' = r' \sin \varphi'$

For $r \gg a$; $|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' = r - r' \sin \theta \sin \varphi'$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i\omega\epsilon a}{2\pi} \frac{e^{ikr}}{r} \int_0^a r' dr' \int_0^{2\pi} d\phi' e^{-ikr' \sin \theta \sin \phi'}$$

Note that : $\frac{1}{2\pi} \int_0^{2\pi} d\phi' e^{-iu \sin \phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r}, \omega) = -i\omega\epsilon a \frac{e^{ikr}}{r} \int_0^a r' dr' J_0(kr' \sin \theta)$$

$$\int_0^w u du J_0(u) = w J_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r}, \omega) = -i\omega\epsilon a^3 \frac{e^{ikr}}{r} \frac{J_1(ka \sin \theta)}{ka \sin \theta}$$

Energy flux : $\mathbf{j}_e = \delta \mathbf{v} p$

Taking time average: $\langle \mathbf{j}_e \rangle = \frac{1}{2} \Re(\delta \mathbf{v} p^*)$

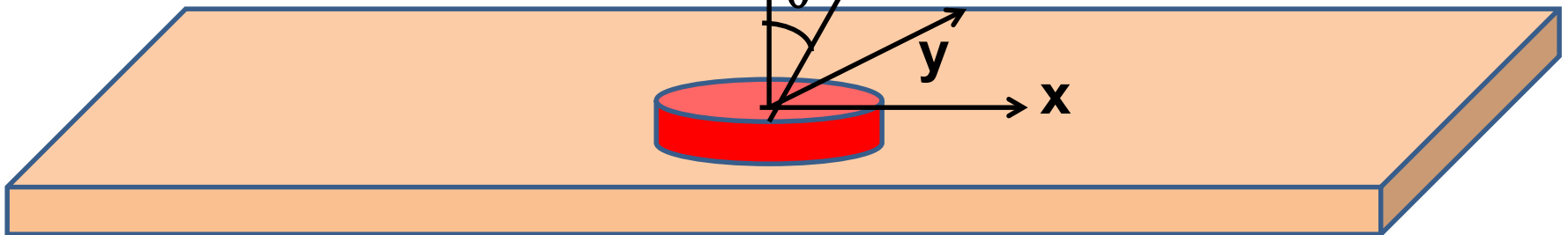
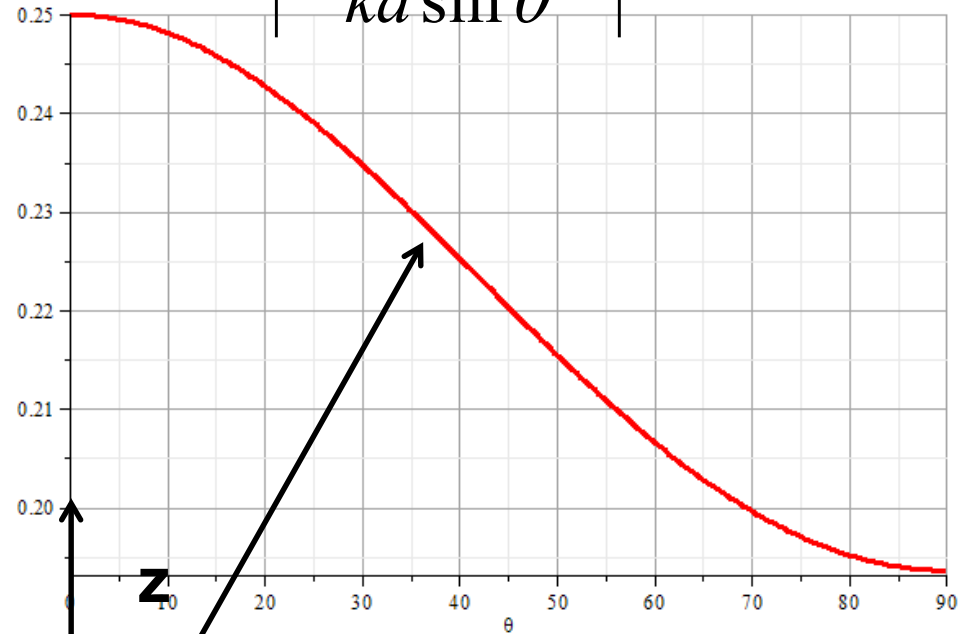
$$= \frac{1}{2} \rho_0 \Re((- \nabla \Phi)(-i\omega\Phi)^*)$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin \theta)}{ka \sin \theta} \right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin \theta)}{ka \sin \theta} \right|^2$$



Scattering of sound waves – for example, from a rigid cylinder

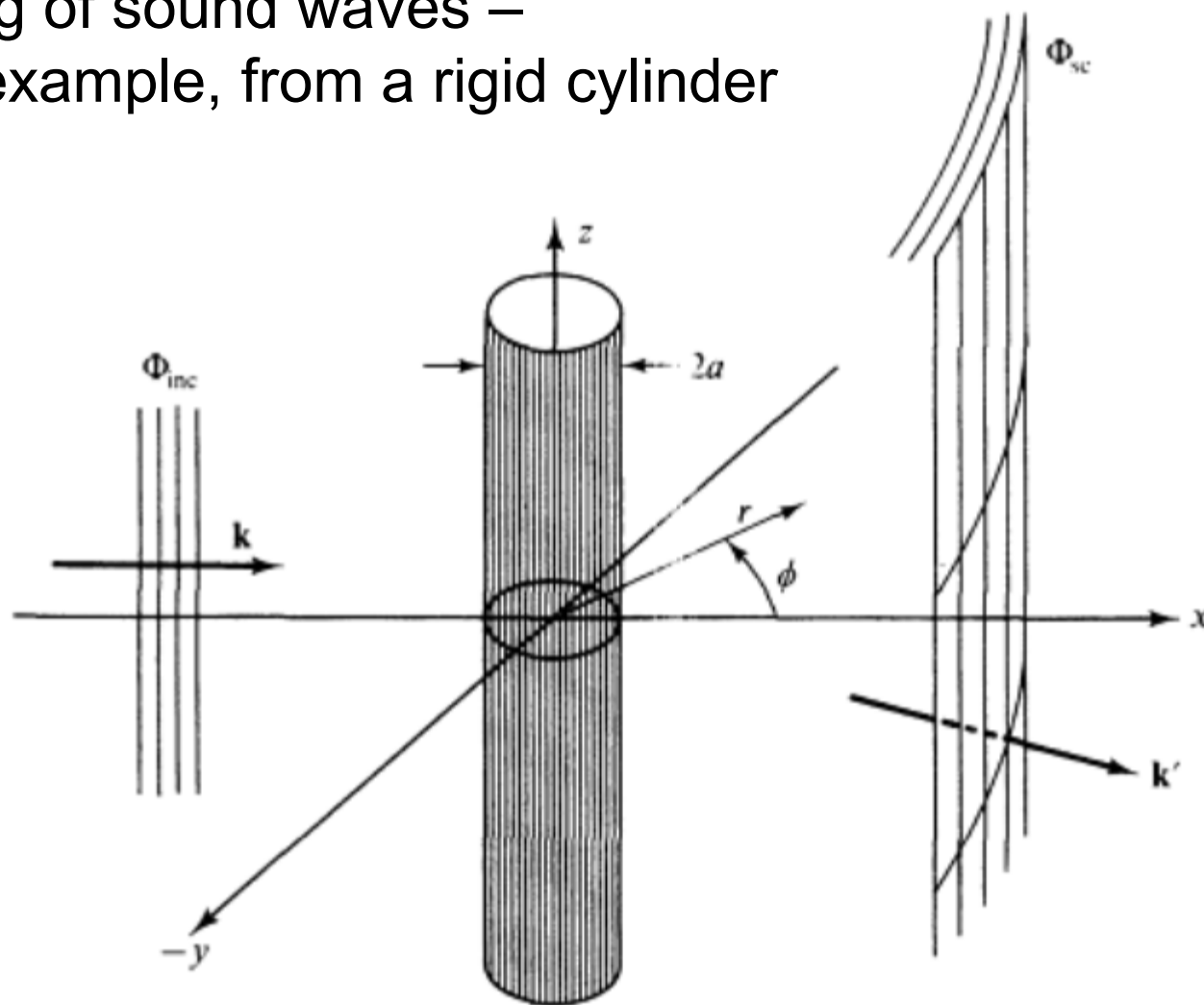


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Scattering of sound waves – for example, from a rigid cylinder

Velocity potential --

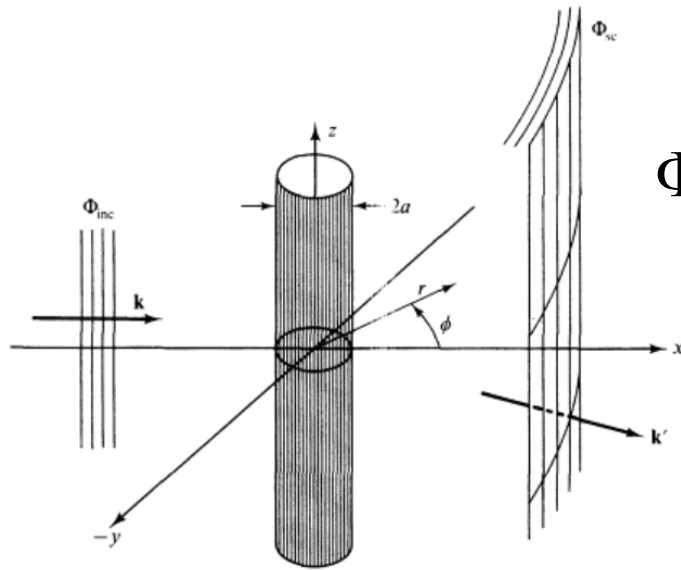
$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}}$$

Helmholtz equation in cylindrical coordinates:

$$(\nabla^2 + k^2)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2 \right) \Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) R_m(r) = 0$



$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} = e^{ikr \cos \phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

$$\text{Boundary condition at } r = a: \quad \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \quad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

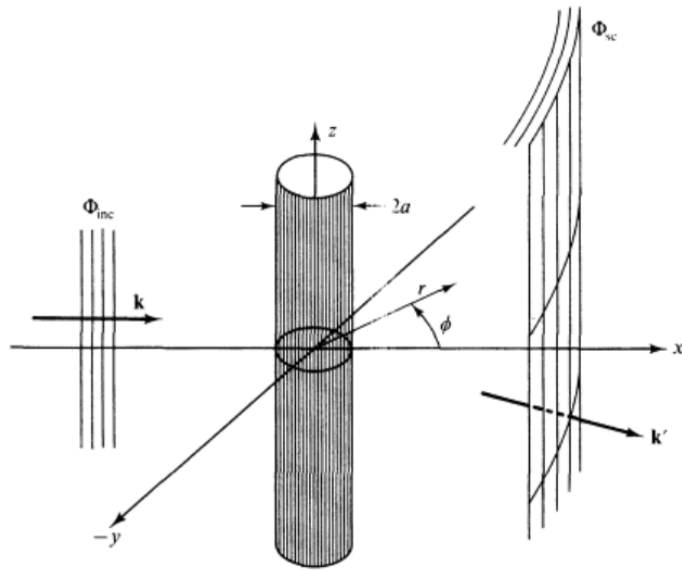


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = - \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr \rightarrow \infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr \rightarrow \infty}{\approx} f(\phi) \sqrt{\frac{1}{r}} e^{ikr} = - \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

$$\Rightarrow f(\phi) = - \sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

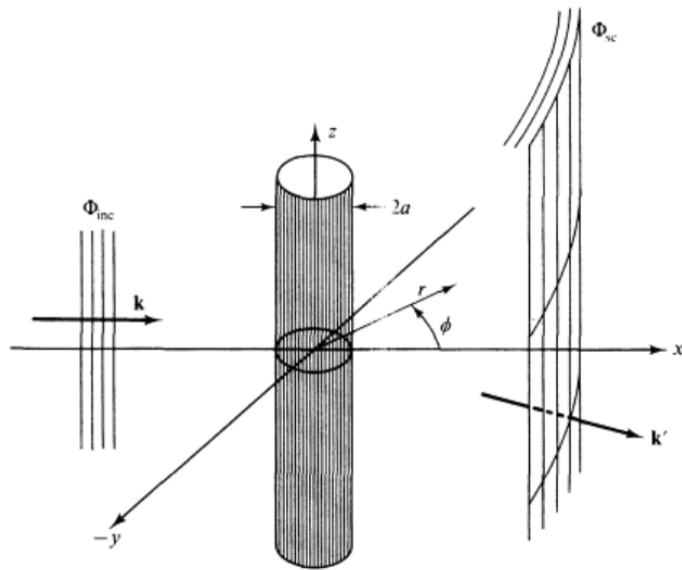


Figure 51.8 Scattering from a rigid cylinder.

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For $ka \ll 1$

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2 \approx \frac{1}{8} \pi k^3 a^4 (1 - 2 \cos \phi)^2$$

