PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online or (occasionally) in Olin 103

Plan for Lecture 31: Chap. 9 of F&W

Wave equation for sound in the linear approximation

1. **Sound generation**
2. **Sound scattering**

In this lecture, we will consider traveling wave solutions to the sound wave equations.

WAKE FOREST UNIVERSITY

Department of Physics

Colloquium

Thursday, 11/4/2020
4 PM online

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"Status and Perspective of Organic Photovoltaic: Is it Ready for Commercialisation?"

The current success of organic semiconductor technology is mainly driven by the development of organic light emitting diodes (OLED), which are now routinely employed in display technologies. In the last decade, however, organic photovoltaics (OPV), leveraging the impressive improvement in device efficiency and stability, have gradually moved from a lab curiosity to a niche market.[1] Their recent success has coincided with the rapid development of effective replacements for the fullerene-based materials that have been prevalent as electron acceptor materials until recently; namely the small molecule nonfullerene acceptors (NFAs).[2] Through strategic design, an acceptor-donor-acceptor (A-D-A) configuration afforded highly absorbing small molecules with tunable energetics, thereby allowing the achievement of record power conversion efficiencies (PCEs) in OPVs. This

Scheduled colloquium for this week hosted by Professor Jurchescu.

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves		
32	Fri, 11/06/2020	Chap. 9	Non linear effects in sound waves		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks		
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids		
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule.

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Note that these sound waves are "longitudinal"

-- the velocity wave direction is along the propagation direction: $\delta\mathbf{v} = -\nabla\Phi = -iA\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$

Review wave equation and plane wave solutions.

Some comments about Monday's lecture

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \qquad \Rightarrow \qquad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \Rightarrow \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential:

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

Note that:

$$\delta\mathbf{v} = -\nabla\Phi(\mathbf{r},t) = -\nabla\tilde{\Phi}(\mathbf{r},t)$$

for $\tilde{\Phi}(\mathbf{r},t) = \Phi(\mathbf{r},t) + \int^{t} dt' K(t')$

$$-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0} = K(t)$$

$$-\frac{\partial\tilde{\Phi}}{\partial t} + \frac{\delta p}{\rho_0} = 0$$

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Review of some details from Monday.

Some comments about Monday's lecture -- continued

Expressing pressure in terms of the density :

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho$$

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow \nabla\left(-\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0}\right) = 0$$

$$\left(-\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0}\right) = K(t) \quad \Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Review continued.

Some comments about Monday's lecture -- continued

Wave equation for air :

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$$\mathbf{v} = -\nabla\Phi$$

Additional relations:

$$\delta p = c^2\delta\rho = \rho_0 \frac{\partial\Phi}{\partial t}$$

$$\Rightarrow \frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2\delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}}\cdot\ \mathbf{V} = \ \hat{\mathbf{n}}\cdot\delta\mathbf{v} = -\ \hat{\mathbf{n}}\cdot\nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial\Phi}{\partial t} = 0$$

Review continued.

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt'\, G(\mathbf{r}-\mathbf{r}',t-t')\, f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}',t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Now think of wave equation with a source. The Green's function is a very powerful tool for solving these problems. We will use similar techniques in solving the wave equation for electromagnetic waves.

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t'-\left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Result that we will derive.

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega$$

First step of derivation using Fourier transform in the time domain.

Derivation of Green's function for wave equation -- continued

$$\text{Define :} \quad \tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t) e^{i\omega t} dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy :

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

Spatial equation for Fourier amplitudes.

Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\widetilde{G}(R,\omega) = 0$$

Solution for isotropic system

Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\widetilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\left(R\widetilde{G}(R,\omega)\right) = 0$$

$$\left(R\widetilde{G}(R,\omega)\right) = A\, e^{ikR} + Be^{-ikR}$$

$$\Rightarrow \; \widetilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

Isotropic solutions continued.

Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

$$\text{Note that :}\quad \nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}$$

A special property of the Laplace operator.

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t')}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Taking the inverse Fourier transform.

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Noting that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} d\omega = \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t-\left(t'\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Details and final result.

➔In order to solve an inhomogenous wave equation with a time harmonic forcing term, we can use the corresponding Green's function:

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)= \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for solving equations with boundary conditions at infinity. For solving problems with surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

It is convenient/important to use the Green's function consistent with the boundary values of the particular system of interest.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}} d^2r$

$$\nabla^2 \tilde{\Phi} + k^2 \tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot \hat{\mathbf{n}} d^2r$$

In order to motivate the use of Green's functions, we consider the famous Green's theorem. Note that these details/derivations will also be discussed when we consider mathematically similar situations for electrodynamic systems.

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) f(\mathbf{r}',\omega)\right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

If the integration volume V includes the point $\mathbf{r} = \mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) f(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

➔extra contributions from boundary

Derivation continued.

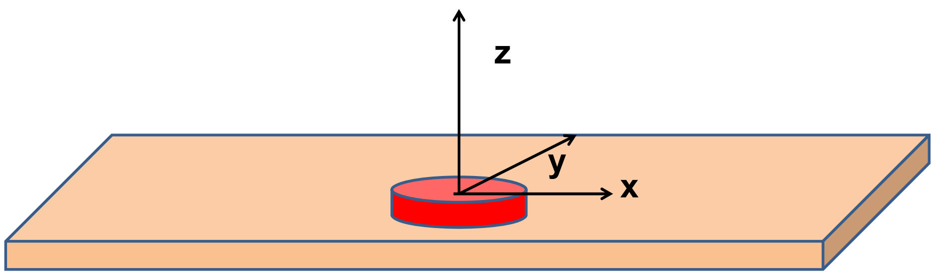
Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$

can be represented as boundary value of $\Phi(\mathbf{r},t)$



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Now consider a simplified model of a sound amplifier where the red cylinder moves up and down in the z direction at a particular frequency omega.

Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

In this case, we need to use a modified Green's function to satisfy the boundary condition at z=0.

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

Some details.

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain: $x' = r'\cos\varphi'$

$y' = r'\sin\varphi'$

For $r >> a$; $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\quad \varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

Changing to more convenient coordinates. Preparing to evaluate the expression far from the moving piston.

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi' e^{-ikr'\sin\theta\sin\phi'}$$

$$\text{Note that:} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\phi' e^{-iu\sin\phi'} = J_0(u)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w udu J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Approximate solution continued. In this approximation, the integral can be evaluated in terms of Bessel functions.

Energy flux: $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average: $\langle \mathbf{j}_e \rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

$$= \frac{1}{2}\rho_0\Re\left(\left(-\nabla\Phi\right)\left(-i\omega\Phi\right)^*\right)$$

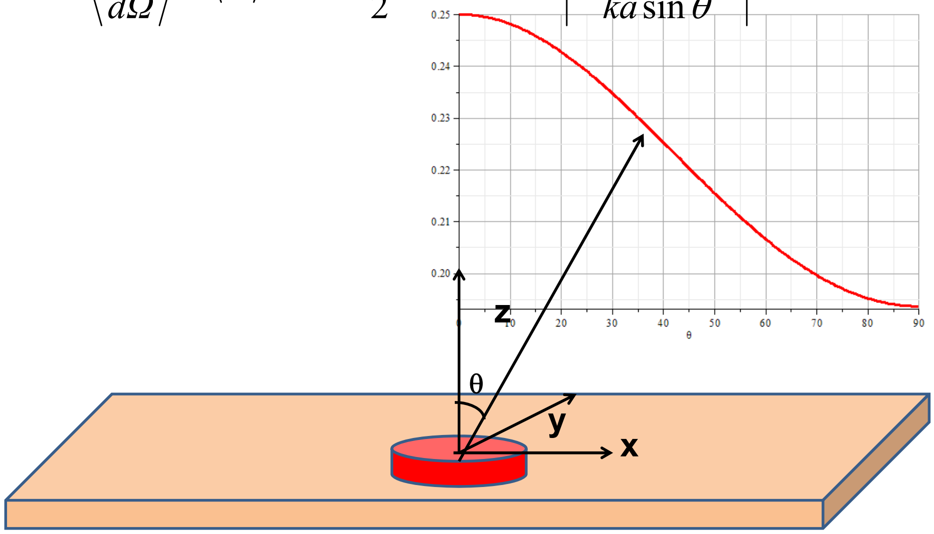
Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Estimating the power of the sound wave in this asymptotic regime.

Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$



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Graph of the power as a function of the polar angle theta.

Scattering of sound waves –
for example, from a rigid cylinder

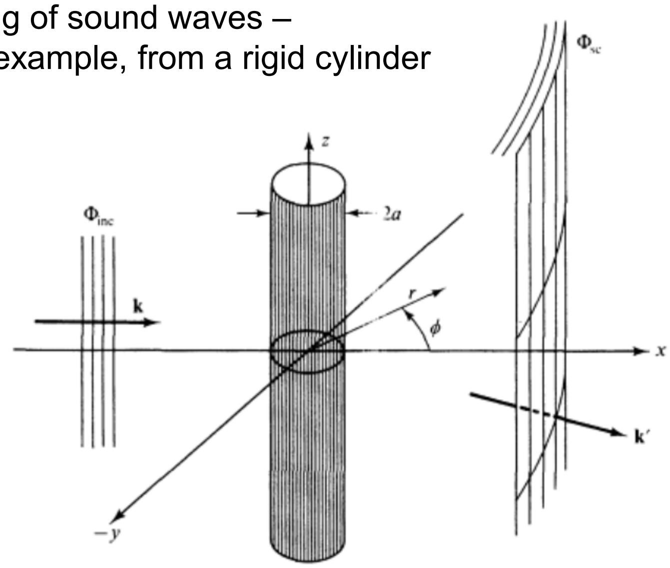

Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Now consider the case of a plane wave of sound, scattering off of a cylindrical object. Can you think of a physical situation for this model?

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$

Analysis of the scattering wave using cylindrical coordinates.

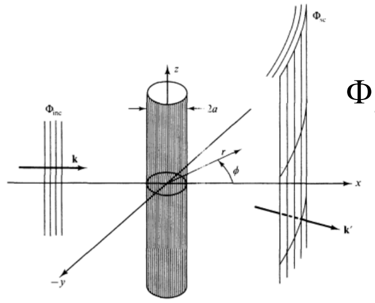

Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave : $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

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In this case we expect a cylindrical wave that can be represented in terms of Bessel and Neumann functions, or more conveniently in terms of Hankel functions H. Satisfying the boundary values on the surface of the scattering cylinder, we find the coefficients of the expression.

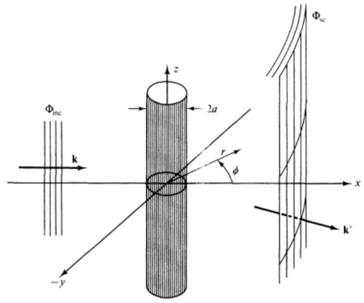


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

Using the asymptotic form of the Hankel functions we can analyze the results further.

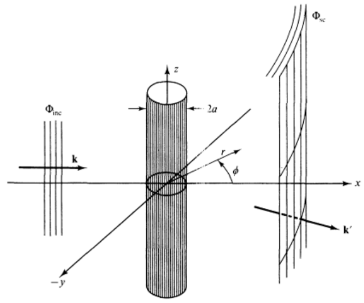
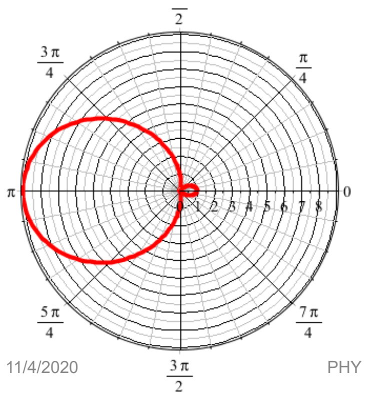


Figure 51.8 Scattering from a rigid cylinder.



$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For *ka << 1*

$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2 \approx \frac{1}{8}\pi k^3 a^4 \left(1 - 2\cos\phi\right)^2$$

Defining the appropriate scattering cross section, we can analyze the results further. For ka<<1 (long wavelengths, low frequencies) we find that most of the sound is scattered backwards from the propagation direction.