

PHY 711 Classical Mechanics and Mathematical Methods

**10-10:50 AM MWF online or (occasionally)
in Olin 103**

Discussion for Lecture 32: Chap. 9 of F&W

**Sound generation and scattering;
non-linear effects**

- 1. Sound generation**
- 2. Sound scattering**
- 3. Introduction to non-linear effects**

Schedule for weekly one-on-one meetings (EST)

Nick – 11 AM Monday

Tim – 9 AM Tuesday

Gao – 9 PM Tuesday

Jeanette – 11 AM Friday

Derek – 12 PM Friday



27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks		
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids		
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Your questions –

From Nick –

1. Can you cover what the extra boundary conditions are representing?
2. Also what is the deal with the bar notation on the r, z ? Why not say $r+r'$?
3. Lastly, can you go over the approximation for $|r-r'|$? I don't quite follow that.

From Gao –

1. Does $u=c+v$ stands for actual wave velocity in nonlinear media?

Recap: Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Solution in terms of Green's function:

$$\Phi(\mathbf{r}, t) = \int d^3 r' \int dt' G(\mathbf{r} - \mathbf{r}', t - t') f(\mathbf{r}', t') +$$

boundary contributions, where

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

Reviewing some of the highlights from last time --
Derivation of Green's function for wave equation -- continued

Define: $\tilde{G}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} G(\mathbf{r}, t) e^{i\omega t} dt$

$$G(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r}, \omega)$ must satisfy:

$$(\nabla^2 + k^2) \tilde{G}(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}') \quad \text{where} \quad k^2 = \frac{\omega^2}{c^2}$$

In our case -- $\tilde{G}(\mathbf{r} - \mathbf{r}', \omega) \Rightarrow \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$

From Green's theorem presented last time --

Fourier transform of velocity potential:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) f(\mathbf{r}', \omega) d^3 r' +$$

Fourier transform
of forcing term.

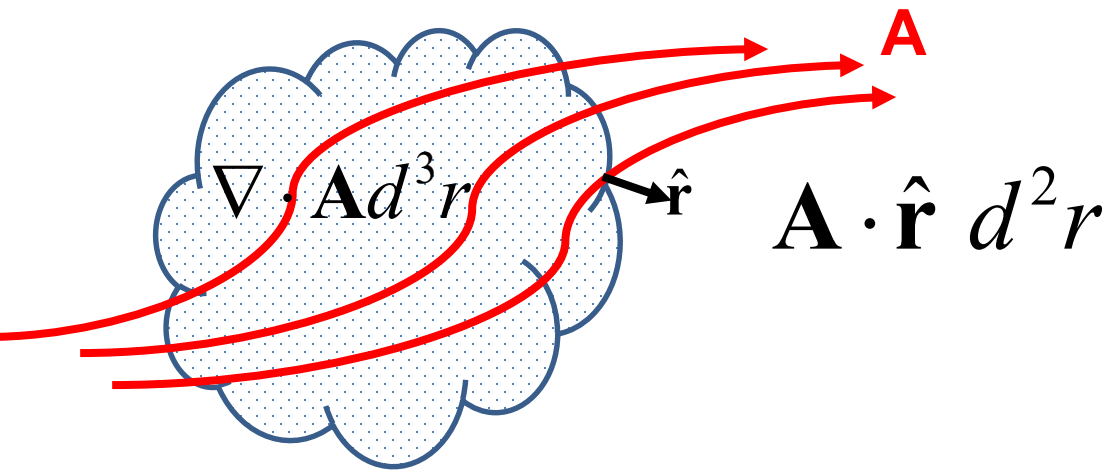
$$\oint_S \left(\tilde{\Phi}(\mathbf{r}', \omega) \nabla' \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla' \tilde{\Phi}(\mathbf{r}', \omega) \right) \cdot \hat{\mathbf{n}}' d^2 r$$

extra contributions from boundary

Your question -- Can you cover what the extra boundary conditions are representing?

Comment – The Green's theorem that we discussed last time is based on the properties of the divergence theorem

$$\int_{\text{Volume}} \nabla \cdot \mathbf{A} d^3r = \int_{\text{Surface}} \mathbf{A} \cdot \hat{\mathbf{r}} d^2r$$



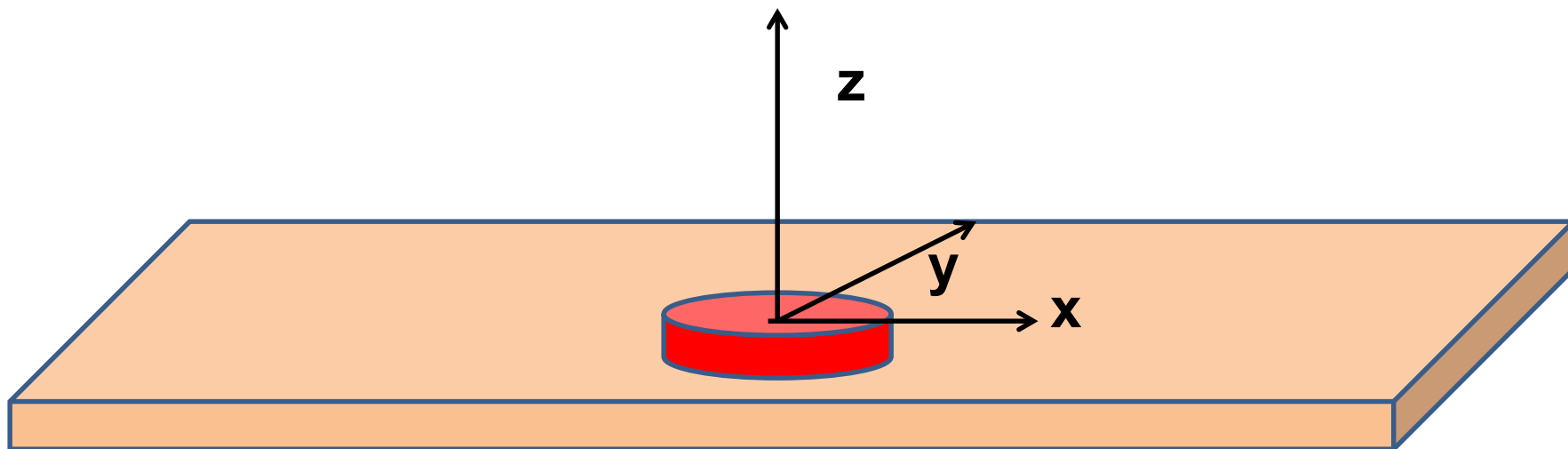
Note: The surface term is important when the analysis is performed in a confining volume. When the analysis involves an infinite volume and $\mathbf{A}$ vanishes at infinity, the surface term does not contribute.

Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Example:

$f(\mathbf{r}, t) \Rightarrow$ time harmonic piston of radius a , amplitude $\varepsilon \hat{\mathbf{z}}$
can be represented as boundary value of $\Phi(\mathbf{r}, t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \tilde{f}(\mathbf{r}', \omega) d^3 r' + \oint_S \left(\tilde{\Phi}(\mathbf{r}', \omega) \nabla' \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla' \tilde{\Phi}(\mathbf{r}', \omega) \right) \cdot \hat{\mathbf{n}} d^2 r'$$

Boundary values for our example: In this case $\tilde{f}(\mathbf{r}, \omega) = 0$

$$\left(\frac{\partial \tilde{\Phi}}{\partial z} \right)_{z=0} = \begin{cases} 0 & \text{for } x^2 + y^2 > a^2 \\ i\omega\epsilon a & \text{for } x^2 + y^2 < a^2 \end{cases} \quad \text{Here our analysis volume is the half space } z > 0.$$

Note: Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

Some more details --

Note: Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

Note that $|\mathbf{r} - \mathbf{r}'| \equiv \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$

$$|\mathbf{r} - \bar{\mathbf{r}}'| \equiv \sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}$$

Fourier transform of velocity potential:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \int_V \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) f(\mathbf{r}', \omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}', \omega) \nabla' \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) - \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \nabla' \tilde{\Phi}(\mathbf{r}', \omega) \right) \cdot \hat{\mathbf{n}}' d^2r'$$

Need this term to vanish at $z' = 0$

Note: Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

Note that $|\mathbf{r} - \mathbf{r}'| \equiv \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$

$$|\mathbf{r} - \bar{\mathbf{r}}'| \equiv \sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}$$

$$\left. \frac{\partial \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega)}{\partial z'} \right|_{z'=0} = 0$$

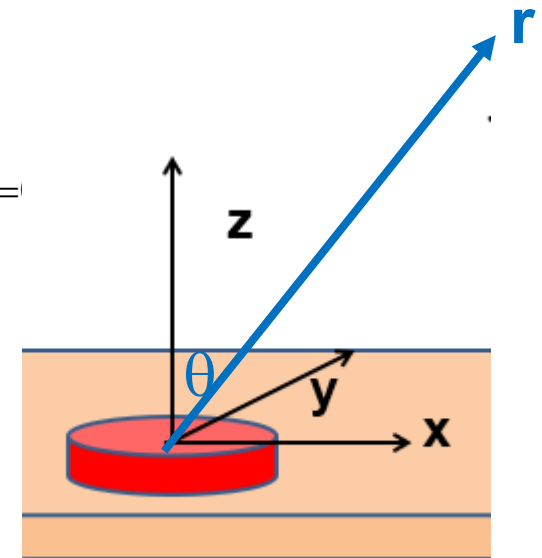
$$\tilde{\Phi}(\mathbf{r}, \omega) = - \oint_{S: z'=0} \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \frac{\partial \tilde{\Phi}(\mathbf{r}', \omega)}{\partial z'} dx' dy'$$

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{e^{ik|\mathbf{r} - \bar{\mathbf{r}}'|}}{4\pi|\mathbf{r} - \bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega)_{z'=0} = \left. \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{2\pi|\mathbf{r} - \mathbf{r}'|} \right|_{z'=0} = \frac{e^{ik\sqrt{(x-x')^2 + (y-y')^2 + z^2}}}{2\pi\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

$$\begin{aligned}\tilde{\Phi}(\mathbf{r}, \omega) &= - \oint_{S: z'=0} \tilde{G}(|\mathbf{r} - \mathbf{r}'|, \omega) \frac{\partial \tilde{\Phi}(\mathbf{r}', \omega)}{\partial z'} dx' dy' \\ &= -i\omega\epsilon a \int_0^a r' dr' \int_0^{2\pi} d\varphi' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{2\pi |\mathbf{r} - \mathbf{r}'|} \Big|_{z'=0}\end{aligned}$$

Integration domain: $x' = r' \cos \varphi'$
 $y' = r' \sin \varphi'$



For $r \gg a$; $|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' = r - r' \sin \theta \sin \varphi'$$

Some details --

$$\text{For } r \gg a; \quad |\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}'$$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin \theta \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' = r - r' \sin \theta \sin \varphi'$$

$$\text{Note that } |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 - 2\mathbf{r} \cdot \mathbf{r}' + r'^2}$$

$$= r \left(1 - \frac{2\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{r'^2}{r^2} \right)^{1/2} \approx r \left(1 - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} \right)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i\omega\epsilon a}{2\pi} \frac{e^{ikr}}{r} \int_0^a r' dr' \int_0^{2\pi} d\phi' e^{-ikr' \sin \theta \sin \phi'}$$

Note that : $\frac{1}{2\pi} \int_0^{2\pi} d\phi' e^{-iu \sin \phi'} = J_0(u)$ From integral from of Bessel functions.

$$\Rightarrow \tilde{\Phi}(\mathbf{r}, \omega) = -i\omega\epsilon a \frac{e^{ikr}}{r} \int_0^a r' dr' J_0(kr' \sin \theta)$$

$$\int_0^w u du J_0(u) = w J_1(w)$$

Another Bessel function identity.

$$\Rightarrow \tilde{\Phi}(\mathbf{r}, \omega) = -i\omega\epsilon a^3 \frac{e^{ikr}}{r} \frac{J_1(ka \sin \theta)}{ka \sin \theta}$$

Analytic result
for $r \rightarrow \infty$

Energy flux : $\mathbf{j}_e = \delta \mathbf{v} p$

Taking time average: $\langle \mathbf{j}_e \rangle = \frac{1}{2} \Re(\delta \mathbf{v} p^*)$

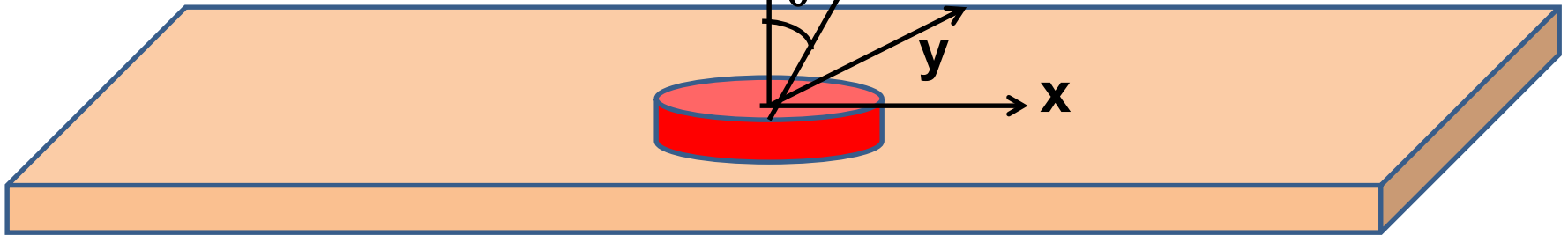
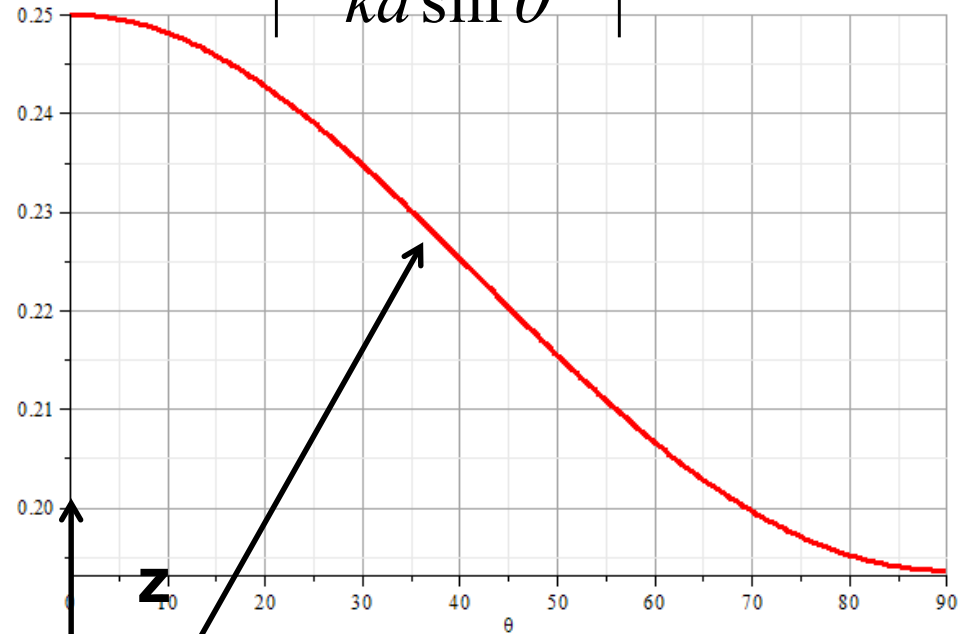
$$= \frac{1}{2} \rho_0 \Re((- \nabla \Phi)(-i\omega\Phi)^*)$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin \theta)}{ka \sin \theta} \right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin \theta)}{ka \sin \theta} \right|^2$$



Scattering of sound waves – for example, from a rigid cylinder

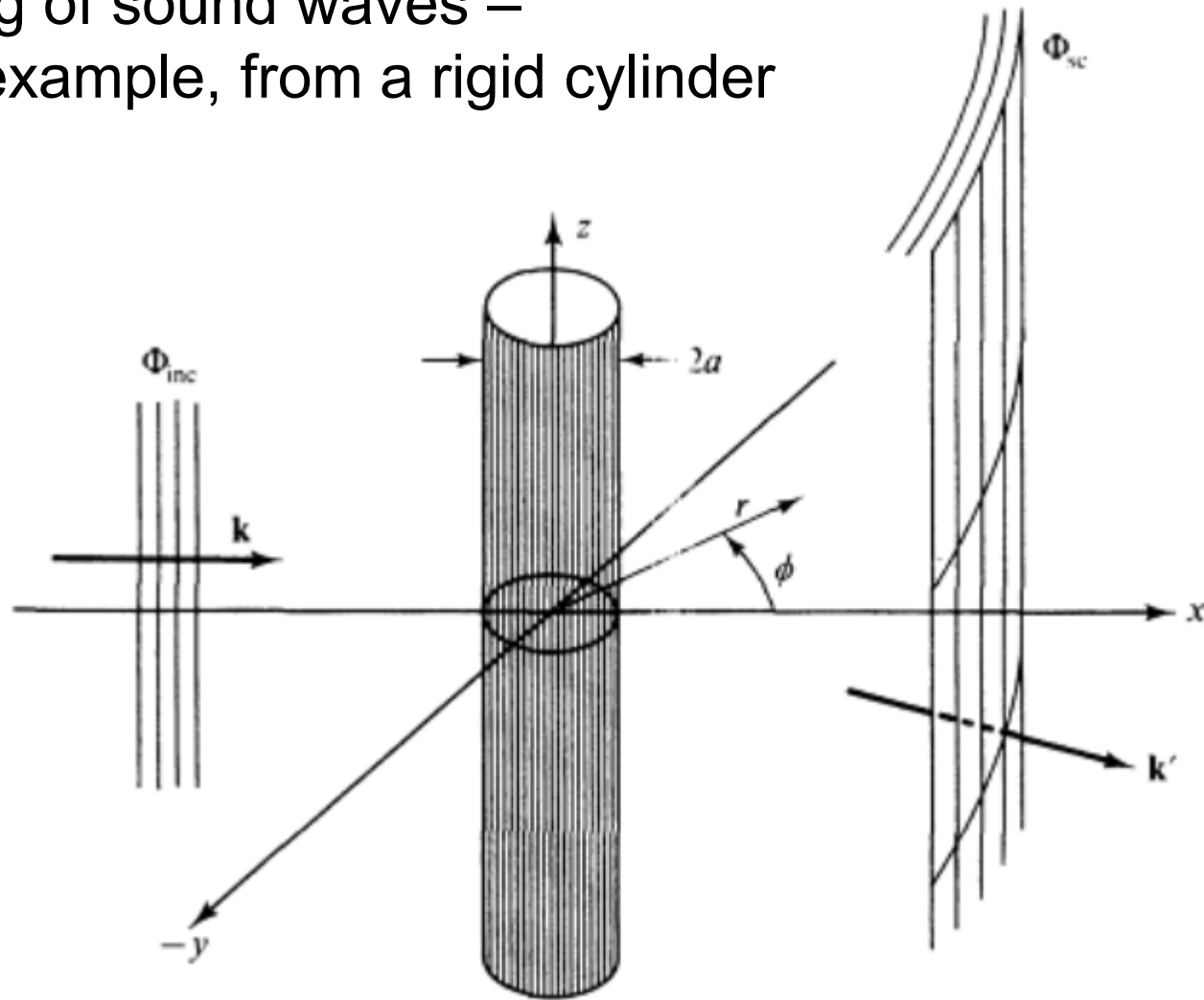


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Example of cylindrical scattering objects --



Suppose a trumpeter is playing near the columns. Maximal scattering occurs when

- Facing toward the column
- Facing away from the column.

Scattering of sound waves – for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}}$$

Helmholtz equation in cylindrical coordinates:

$$(\nabla^2 + k^2)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2 \right) \Phi(\mathbf{r})$$

$$\text{Assume: } \Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$$

$$\text{where} \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) R_m(r) = 0$$

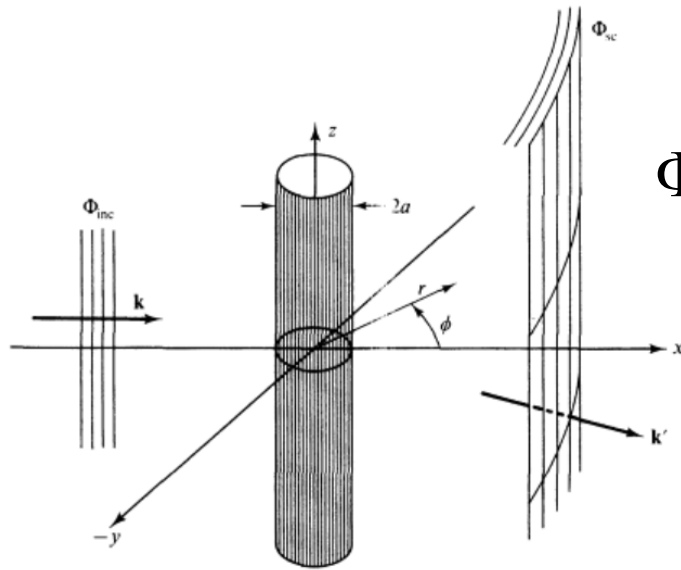


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} = e^{ikr \cos \phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

$$\text{Boundary condition at } r = a: \quad \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \quad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

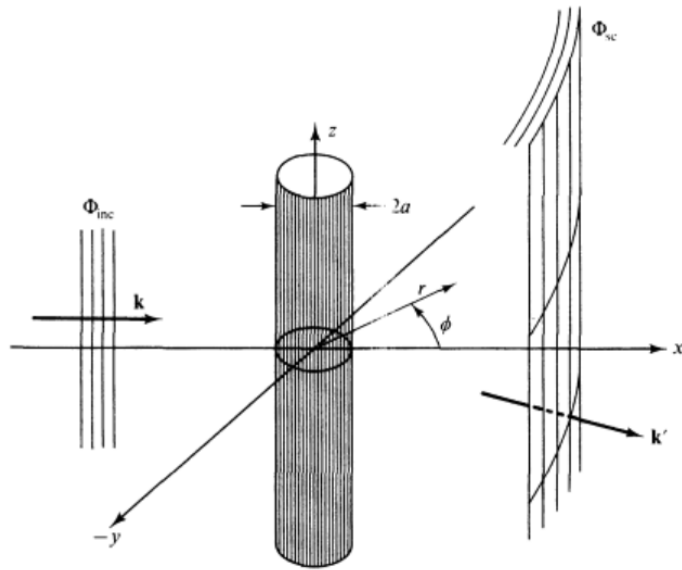


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = - \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr \rightarrow \infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr \rightarrow \infty}{\approx} f(\phi) \sqrt{\frac{1}{r}} e^{ikr} = - \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

$$\Rightarrow f(\phi) = - \sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

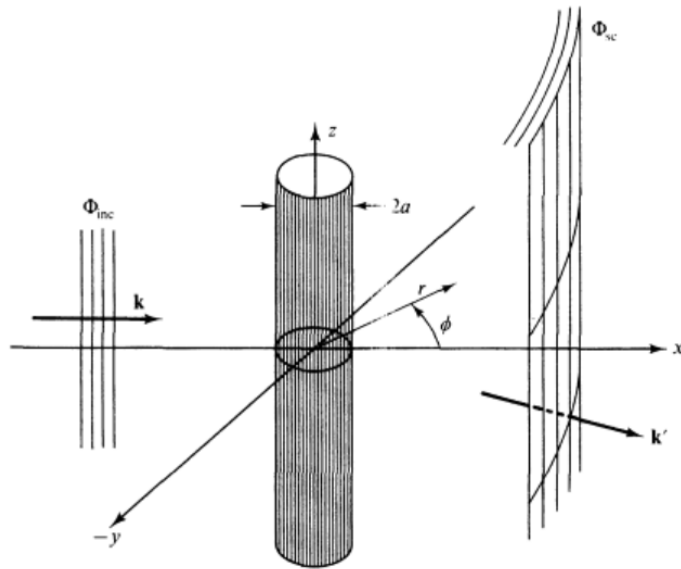


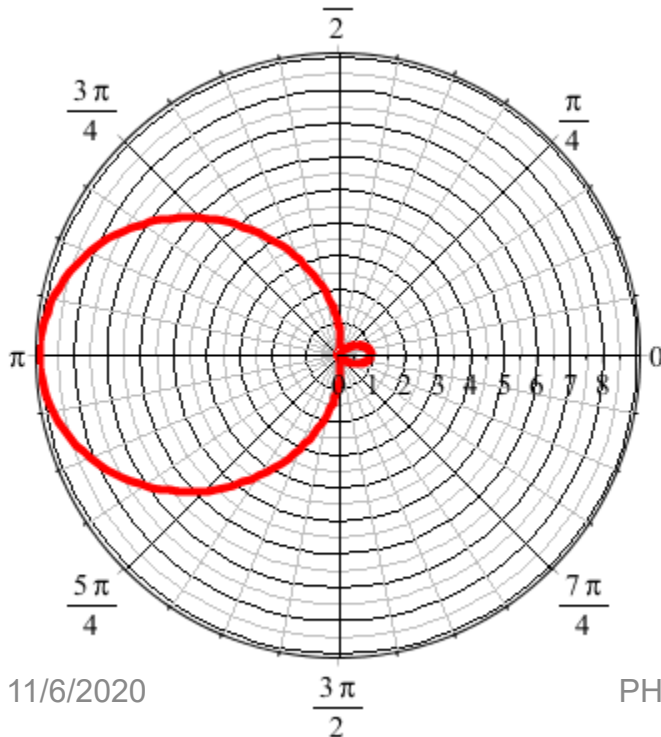
Figure 51.8 Scattering from a rigid cylinder.

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For $ka \ll 1$

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2 \approx \frac{1}{8} \pi k^3 a^4 (1 - 2 \cos \phi)^2$$

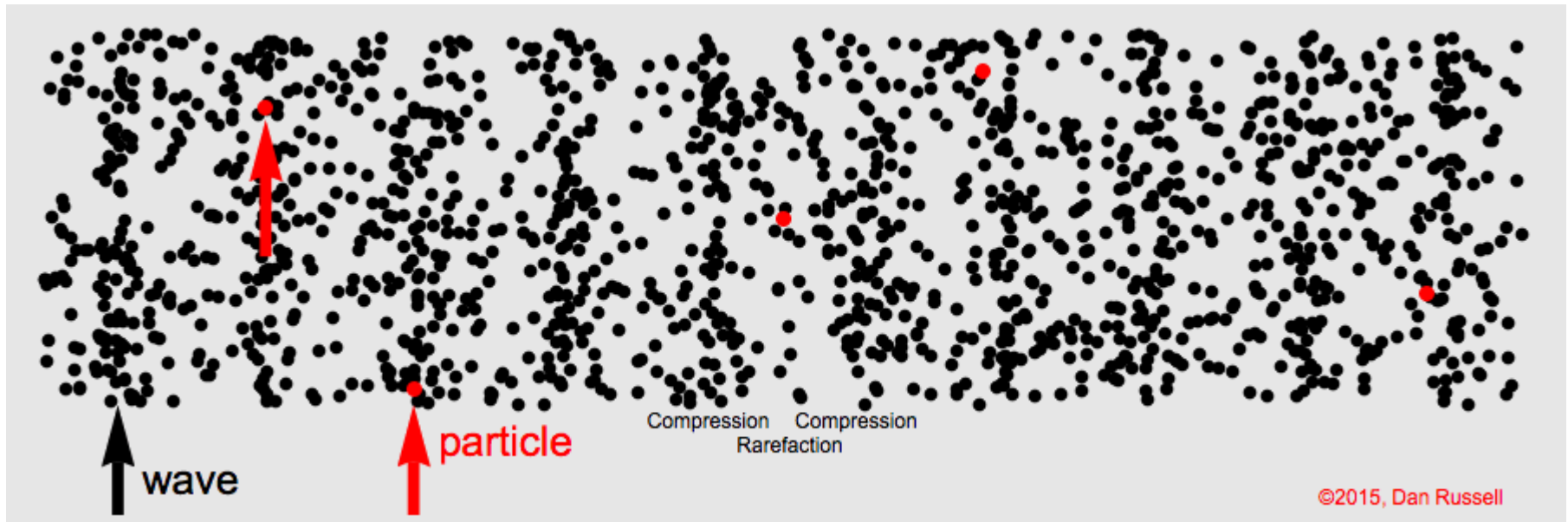


Note: As explained in the textbook, this “cross section” has the units of length, so that it could be interpreted as a scattering width because of the cylindrical geometry.

Visualization of longitudinal wave motion

From the website:

<https://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html>



Now consider some non-linear effects

Effects of nonlinearities in fluid equations

-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

$$\text{Continuity equation : } \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Assume spatial variation confined to x direction ;

assume that $\mathbf{v} = v \hat{\mathbf{x}}$ and $\mathbf{f}_{\text{applied}} = 0$.

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ : $p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x} \equiv c^2(\rho) \frac{\partial \rho}{\partial x} \quad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \quad p = p_0 \left(\frac{\rho}{\rho_0} \right)^\gamma$$

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1} \quad \text{where} \quad c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial t} + v \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} = 0$$

Some more algebra :

$$\text{From Euler equation : } \frac{\partial v}{\partial \rho} \left(\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} \right) + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$$

$$\text{From continuity equation : } \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = -\rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x}$$

$$\text{Combined equation : } \frac{\partial v}{\partial \rho} \left(-\rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} \right) + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \left(\frac{\partial v}{\partial \rho} \right)^2 = \frac{c^2(\rho)}{\rho^2} \quad \frac{\partial v}{\partial \rho} = \pm \frac{c}{\rho}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1}$ $c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \quad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0} \right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma-1} \left(\left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right)$$

$$\Rightarrow c = c_0 \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2}$$

Summary :

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1}$ $c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} \quad v = \pm \frac{2c_0}{\gamma-1} \left(\left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right)$$

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0 \left(\frac{\gamma + 1}{\gamma - 1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$$

Assume: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0 \left(\frac{\gamma + 1}{\gamma - 1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

Solution in linear approximation:

$$u = v + c \approx v_0 + c_0 = c_0 \left(\frac{\gamma + 1}{\gamma - 1} - \frac{2}{\gamma - 1} \right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_w)$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0 \left(\frac{\gamma + 1}{\gamma - 1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

$$u = c_0 \left(\frac{\gamma + 1}{\gamma - 1} \left(1 + s(x - ut) \right)^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

Visualization continued:

$$u = c_0 \left(\frac{\gamma + 1}{\gamma - 1} (1 + s(x - ut))^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

Plot $s(x - ut)$ for fixed t , as a function of x :

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w, t)$$

$$u(w) = c_0 \left(\frac{\gamma + 1}{\gamma - 1} (1 + s(w))^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

Parametric equations:

plot $s(w)$ vs $x(w, t)$ for range of w at each t

Summary

$$\frac{\partial \rho}{\partial t} + u(\rho) \frac{\partial \rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 (1 + s(x - u(\rho)t))$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0 \left(\frac{\gamma + 1}{\gamma - 1} (1 + s(x - ut))^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$

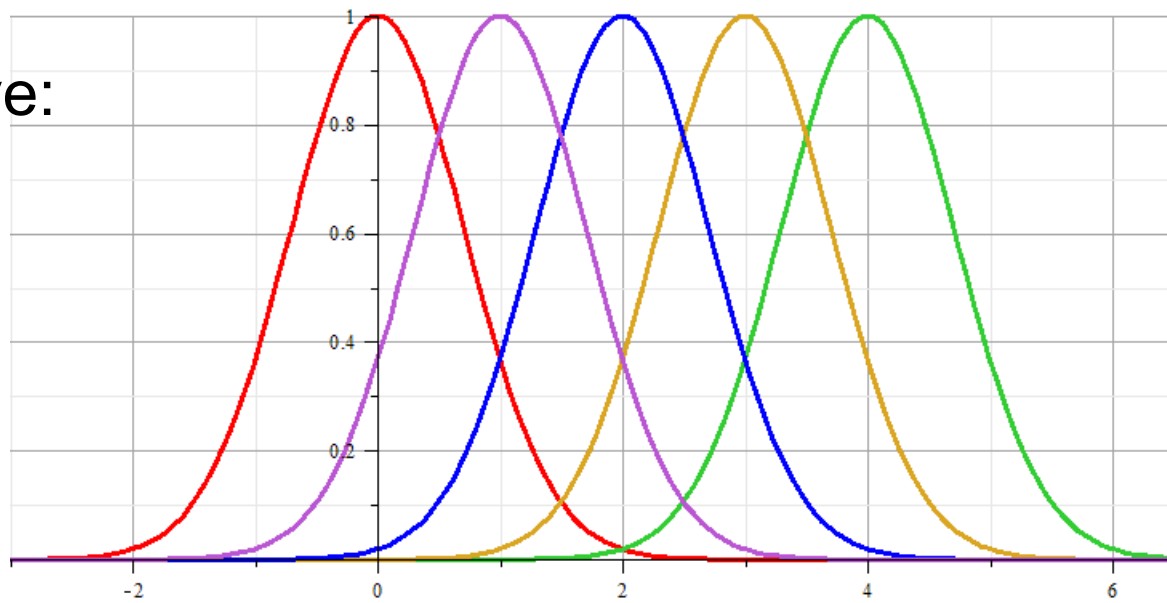
Plot $s(x - ut)$ for fixed t , as a function of x :

Let $w = x - ut \Rightarrow x = w + ut = w + u(w)t \equiv x(w, t)$

$$u(w) = c_0 \left(\frac{\gamma + 1}{\gamma - 1} (1 + s(w))^{(\gamma-1)/2} - \frac{2}{\gamma - 1} \right)$$

Parametric equations: plot $s(w)$ vs $x(w, t)$ for range of w

Linear wave:



Non-linear wave:

