PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Plan for Lecture 33: Chap. 9 of F&W

Wave equation for sound beyond the linear approximation

1. Non-linear effects in sound waves
2. Shock wave analysis

In today's lecture, we will review the basic hydrodynamic equations, specializing to one dimensional traveling waves, and analyzing some of the effects of nonlinearities.

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids		
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Updated schedule

PHY 711 -- Assignment #21

Nov. 09, 2020

Finish reading Chapter 9 in **Fetter & Walecka**.

1. In class, we discussed how to visualize the non-linear behavior of an adiabatic ideal gas with parameter γ. Using Maple or Mathematica or other software and using a parametric plot formalism, create an animated gif file to show the traveling waveform $s(w)$, where s is a shape of your choice and $w=x-u(s(w))t$. You will also need to choose the value of γ as well.

Homework for Wednesday.

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$

Assume spatial variation confined to x direction ;
assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0.$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Starting with the basic hydrodynamic equations and expressing them for the one spatial dimensional case (along the x axis).

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas: $\frac{dp}{p} = \gamma\frac{d\rho}{\rho} \qquad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where} \quad c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

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The equations depend on density (rho), pressure (p), and velocity (v). We need to express all of the results in terms of density (rho). Here we further assume an ideal gas and express the results in terms of the parameter gamma.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Here we also express the velocity in terms of the density rho.

Some more algebra :

From Euler equation : $$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

From continuity equation : $$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$$

Combined equation : $$\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+\left(v\pm c\right)\frac{\partial \rho}{\partial x}=0$$

Combining the results from the two coupled equations, we arrive at a first order differential equation for rho(x,t).

Assuming adiabatic process: $c^2 = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm\frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho}\left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2}\frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

$$\Rightarrow c = c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

Using the results of an adiabatic ideal gas, we can find the explicit functional forms for v and c as they depend on the density rho.

Summary:

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process: $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm \frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Summary of previous slides.

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Now we can analyze the results. We introduce the parameter u(rho) as v(rho)+c(rho) for the + solution; representing a traveling wave.

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

$$\text{Assume: } \quad \rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

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For this case, and evaluating the expressions for the adiabatic ideal gas, we obtain the given equations. Checking for the result in the linear limit, we retrieve the expected behavior.

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

In order to visualize what the results mean, we can use Maple or Mathematica in the parametric plot mode.

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations:

plot $s(w)$ vs $x(w,t)$ for range of w at each t

Details of the parametric formulation

Summary

$$\frac{\partial \rho}{\partial t} + u(\rho)\frac{\partial \rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0\left(1 + s(x - u(\rho)t)\right)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

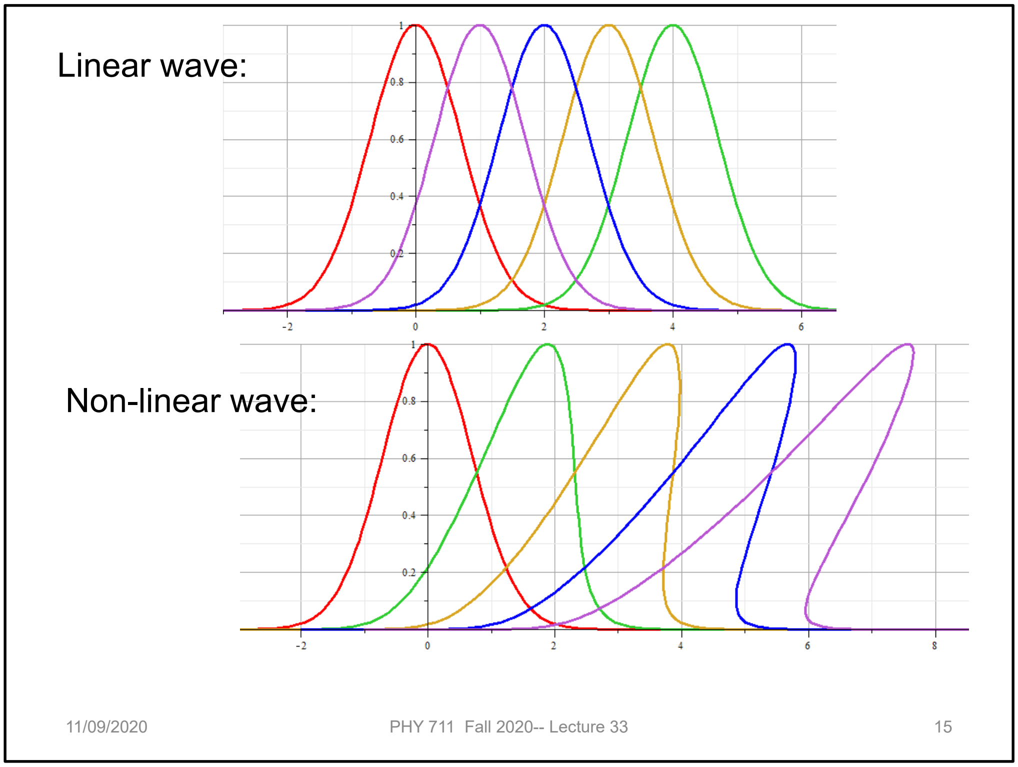
Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut \quad \Rightarrow \quad x = w + ut = w + u(w)t \equiv x(w,t)$

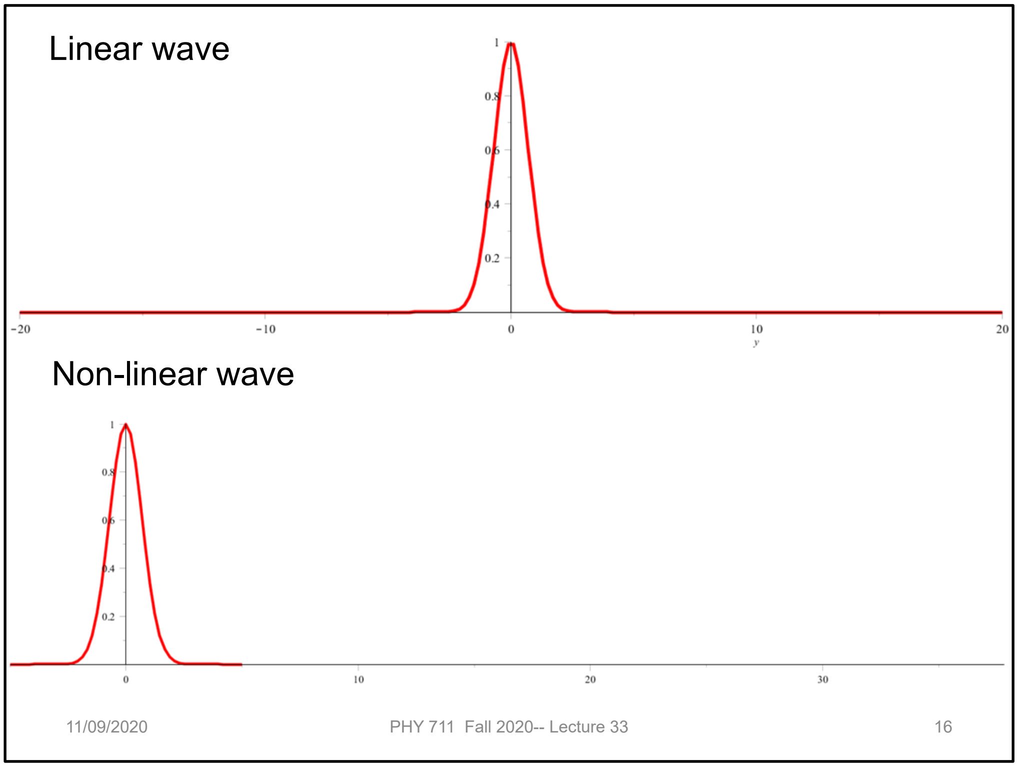
$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

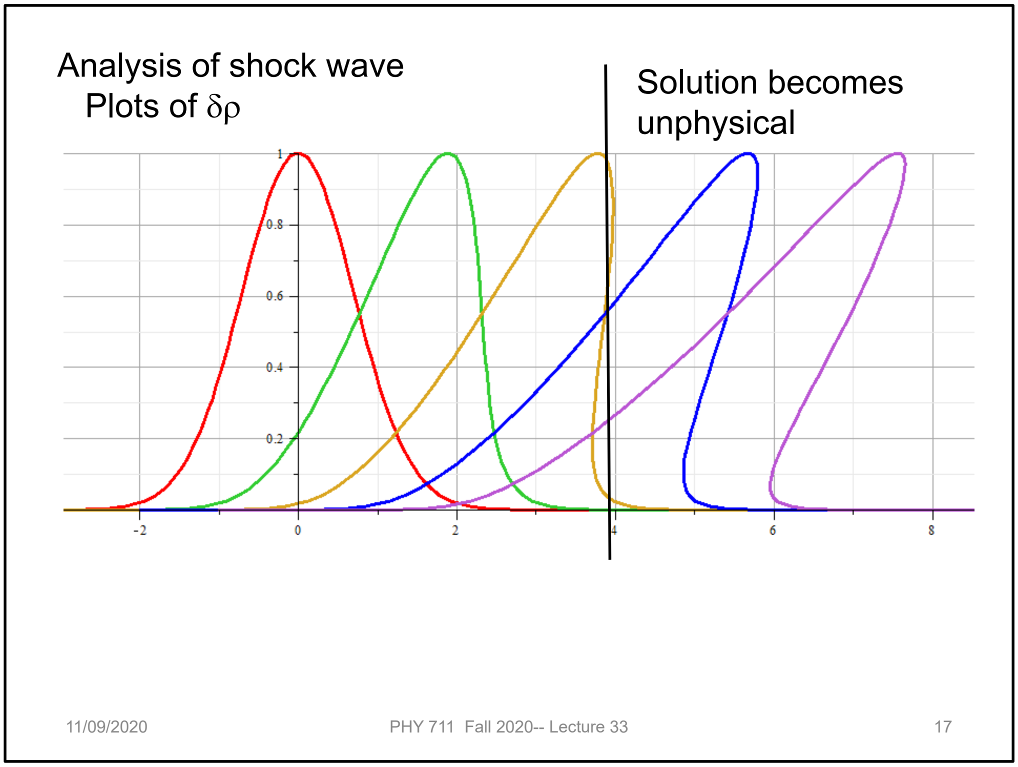
Summary of results.



Snap shots of solution for an initial Gaussian waveform for the linear and non-linear solutions.

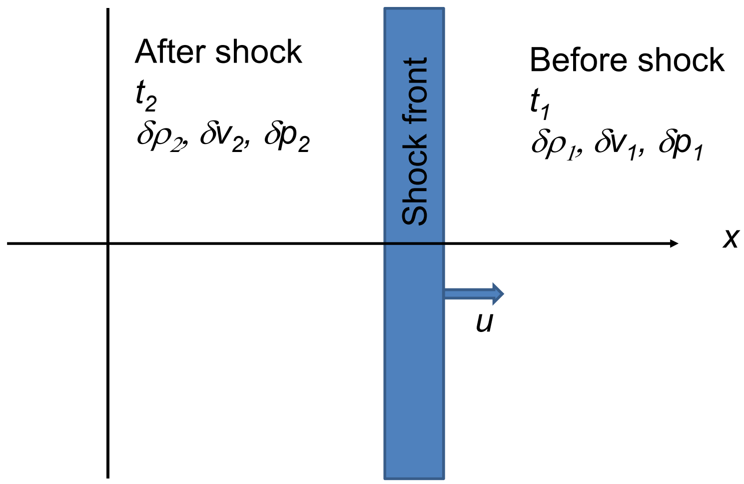


Animations from Maple.



Note that the vertical axis represents the longitudinal wave displacement. When this displacement becomes multivalued for a given coordinate x as shown, the solution becomes unphysical. At this point we need to consider the analysis in a different way.

Analysis of shock wave -- continued



Note that in this case u is assumed to be a given parameter of the system.

Your textbook discusses the shock wave analysis. Here we assume that there is a region (blue) where the analysis fails, but assumes that we can properly analyze the physics before and after the shock. The notation given here is similar to that given in your text.

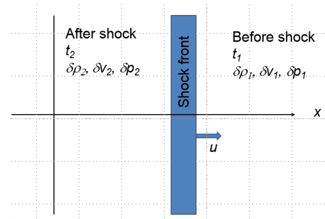
Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume $\rho(x,t) = \rho(x - ut)$

$p(x,t) = p(x - ut)$

$v(x,t) = v(x - ut)$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 = \frac{\partial(\rho v - \rho u)}{\partial x} \qquad \Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

Conservation of energy and momentum:

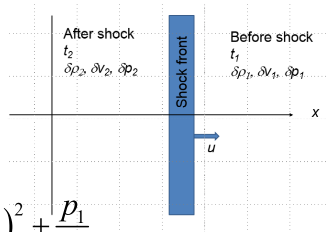
$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Some of the details of the analysis before and after the shock event.

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2



Summary of equations

$$\Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Assume that within each regions (1 & 2), the ideal gas equations apply

$$\epsilon_1 + \frac{p_1}{\rho_1} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} \qquad \epsilon_2 + \frac{p_2}{\rho_2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}$$

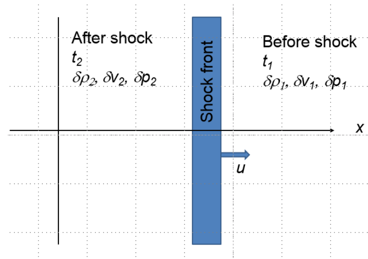
It follows that $\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}(v_2 - u)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - u)^2$

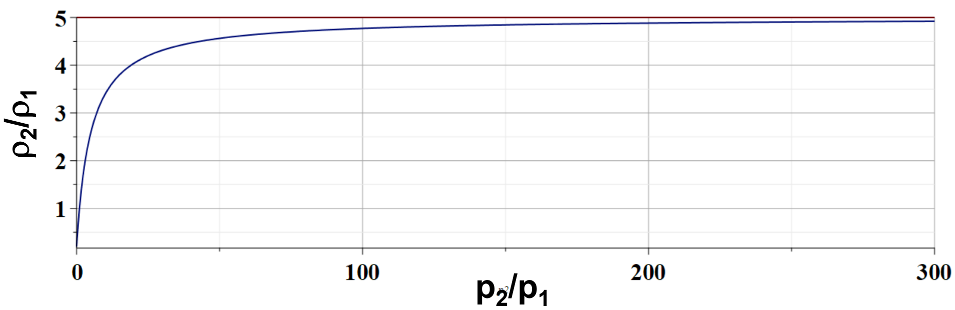
Analyzing the equations.

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$





Analyzing ratio of the density after and before the shock wave.

Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Internal energy density: $\varepsilon = \dfrac{p}{(\gamma-1)\rho} = C_V T$

First law of thermo: $d\varepsilon = Tds - pd\left(\dfrac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) - pd\left(\frac{1}{\rho}\right)\right) = C_V d\ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = C_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < C_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$

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Analyzing the entropy before and after the shock wave. In general, many more relationships can be analyzed. Consult your textbook for more details.