PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Discussion for Lecture 34:

Chapter 10 in F & W: Surface waves

1. Water waves in a channel
2. Wave-like solutions; wave speed

WAKE FOREST UNIVERSITY

Department of Physics

Colloquium

Thursday, Nov. 12, 2020
4 PM

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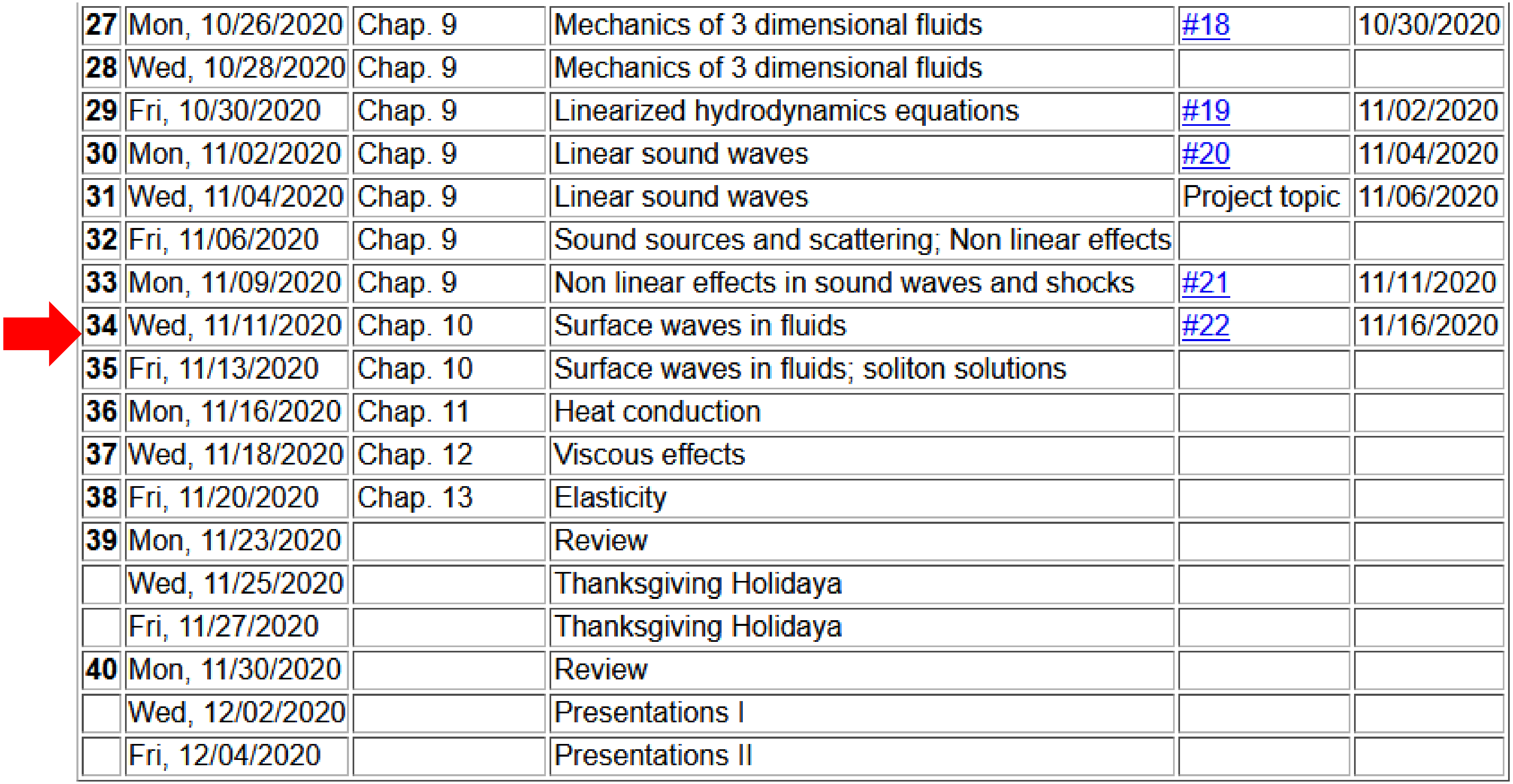
"Shedding Light on the Gaseous Halos of Galaxies in the Early Universe"

Simulations predict that galaxy evolution is regulated by the accretion, expulsion, cooling, and heating of gas in the halos that surround galaxies out to hundreds of kiloparsecs. Distant quasars can be used to backlight and detect circumgalactic gas in absorption, and to date, tens of thousands of intervening absorbers have been detected in large spectroscopic quasar surveys.



27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

PHY 711 -- Assignment #22

Nov. 11, 2020

Start reading Chapter 10 in **Fetter & Walecka**.

1. Work Problem 10.3 at the end of Chapter 10 in **Fetter and Walecka**.

Schedule for weekly one-on-one meetings (EST)

Nick – 11 AM Monday
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Your questions –

From Gao –

1. How does this equation [arise]?

From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

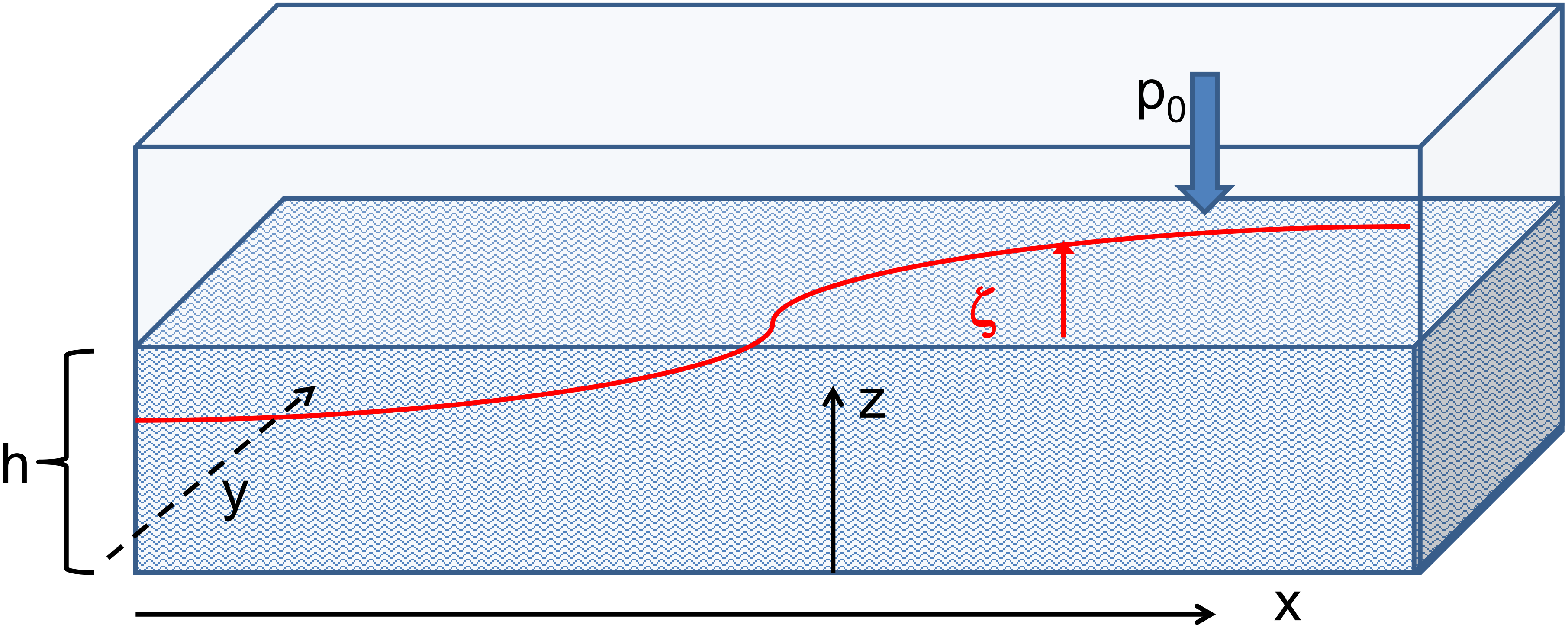
Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)

Atmospheric pressure is in equilibrium with the surface of water

Pressure at a height z above the bottom where the surface is at a height $h+\zeta$:

$$p(z) = \begin{cases} p_0 + \rho g\left(h+\zeta - z\right) & \text{For } z \leq h+\zeta \\ p_0 & \text{For } z > h+\zeta \end{cases}$$

Here ρ represents density of water



Why do we not consider ρ_{air} in this analysis?

a. Because it is a reasonable approximation
b. Because it simplifies the analysis

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\qquad \Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\left(\zeta(x,y,t) + h - z\right)$$

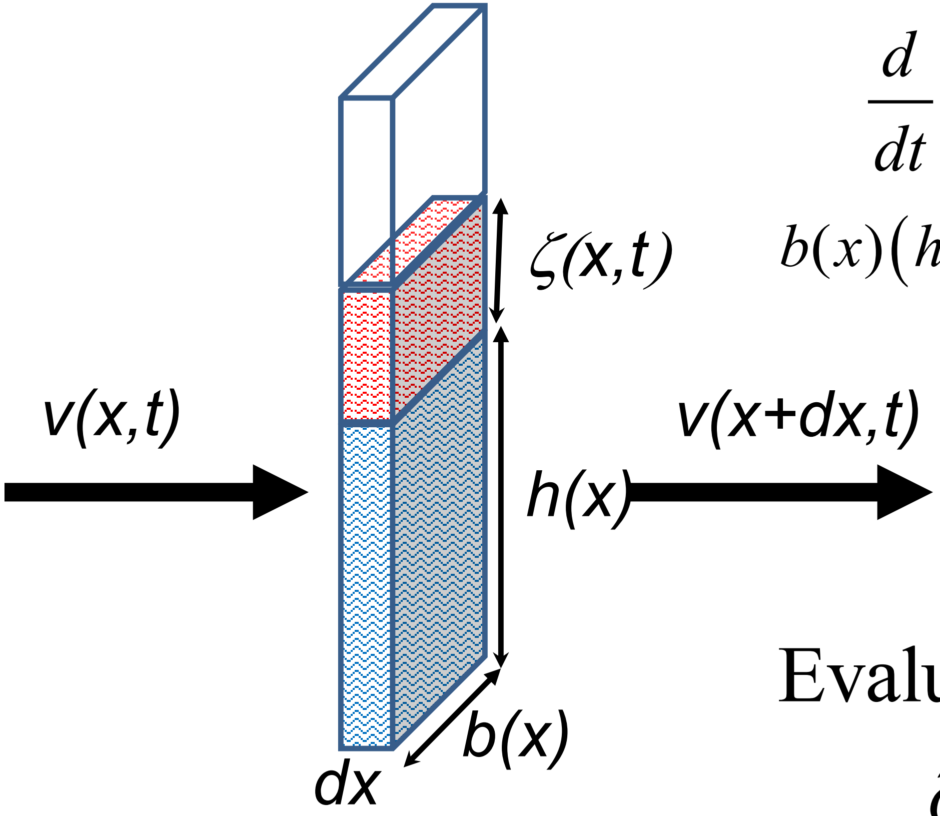
within the water

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

Consider a surface $\zeta(x,t)$ wave moving in the x-direction in a channel of width $b(x)$ and height $h(x)$:



Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

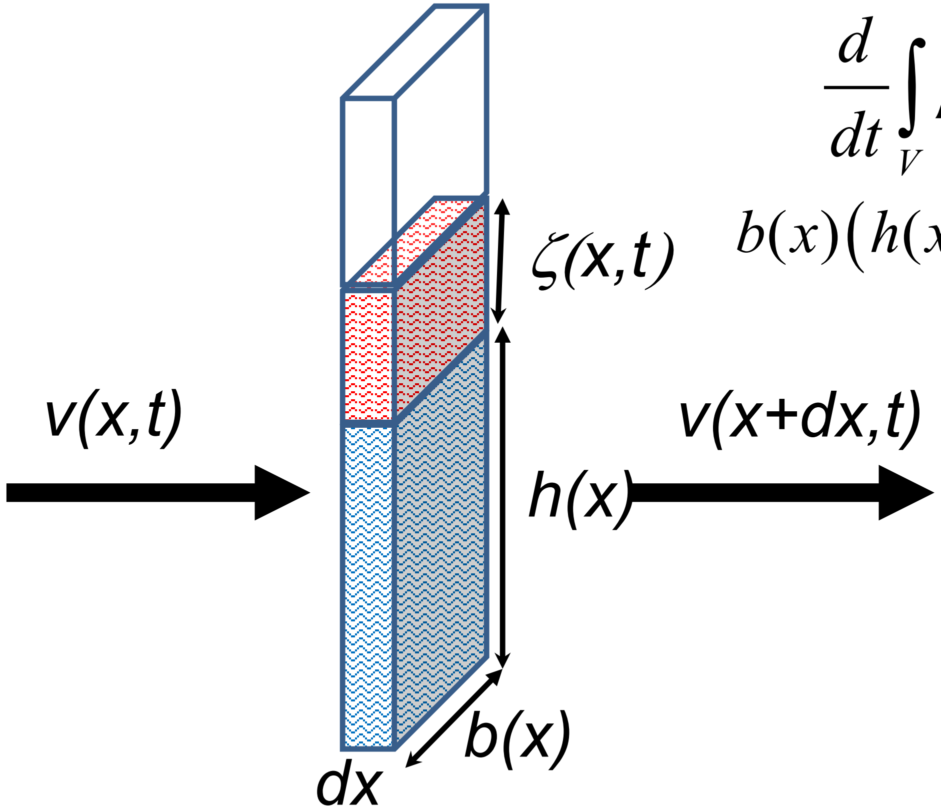
$$b(x)\big(h(x)+\zeta(x,t)\big)\hat{\mathbf{x}}$$

$$b(x)\big(h(x)+\zeta(x,t)\big)dx$$

Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Some details



Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v} \cdot d\mathbf{A} = 0$$

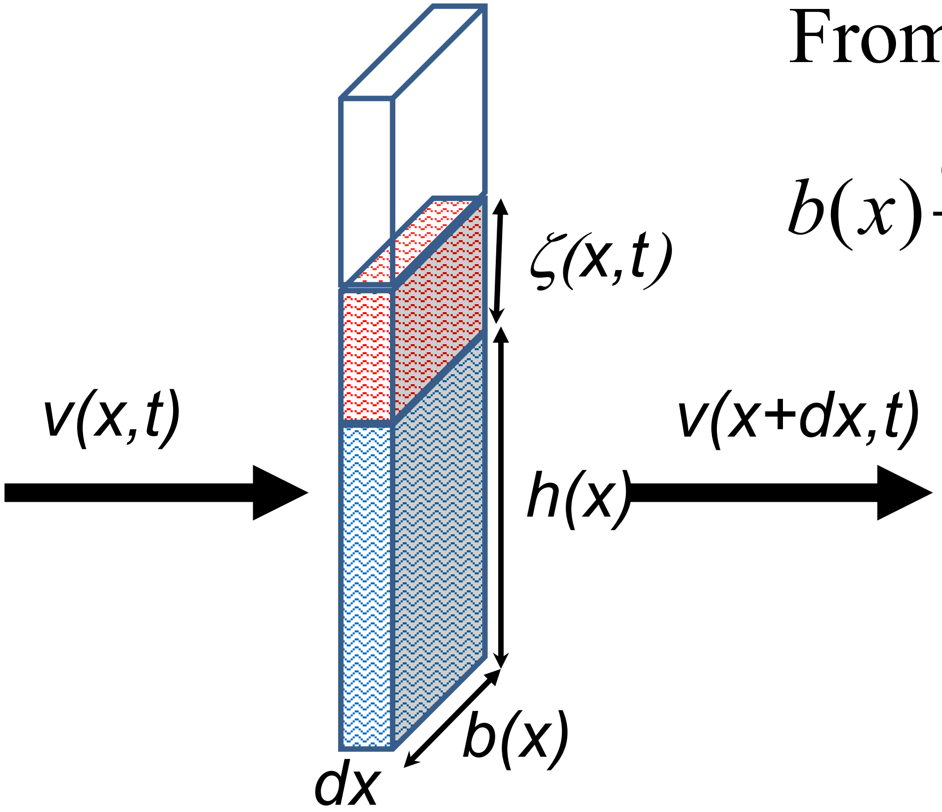
$$b(x)\left(h(x)+\zeta(x,t)\right)dx$$

$$b(x)\left(h(x)+\zeta(x,t)\right)\hat{\mathbf{x}}$$

Here, we are assuming that ρ is constant

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v} \cdot d\mathbf{A} = \rho \int b(x)\frac{\partial \zeta}{\partial t}dx + \rho \int \frac{\partial}{\partial x}\left(b(x)(h(x)+\zeta(x,t))v(x,t)\right)dx = 0$$

$$\Rightarrow b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$



From continuity condition:

$$b(x)\frac{\partial\zeta}{\partial t} = -\frac{\partial}{\partial x}\Big(h(x)b(x)v(x,t)\Big)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0\frac{\partial\zeta}{\partial t} = -\frac{\partial}{\partial x}\Big((\kappa x)b_0 v(x,t)\Big)$$

$$\frac{\partial\zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial\zeta}{\partial x}$$

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du}\right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$

Example continued

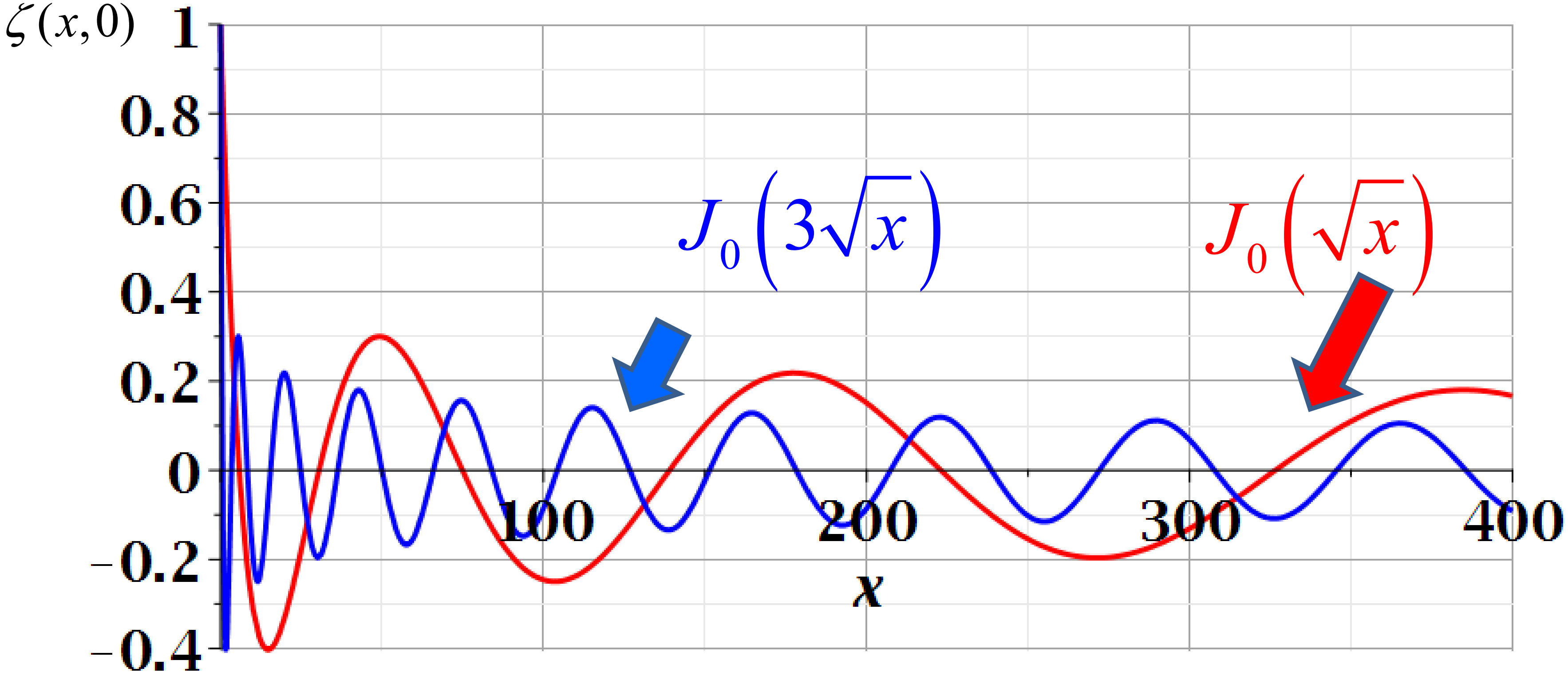
$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left(\frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

$$\Rightarrow \zeta(x,t) = C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

Check:

$$-\omega^2 C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left(\frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

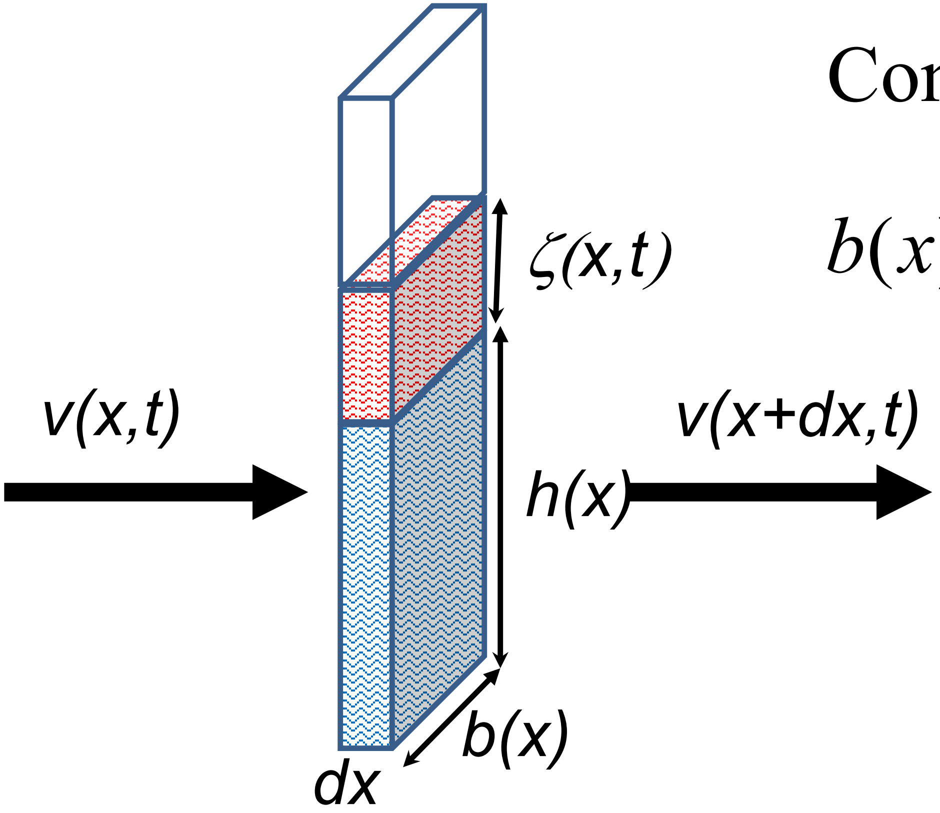
$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$



Imagine watching the waves at a beach – can you visualize the configuration for the surface wave pattern to approximation this situation?

a. Long flat beach
b. Beach in which average water level increases
c. Beach in which average water level decreases

A simplier example:



Continuity condition:

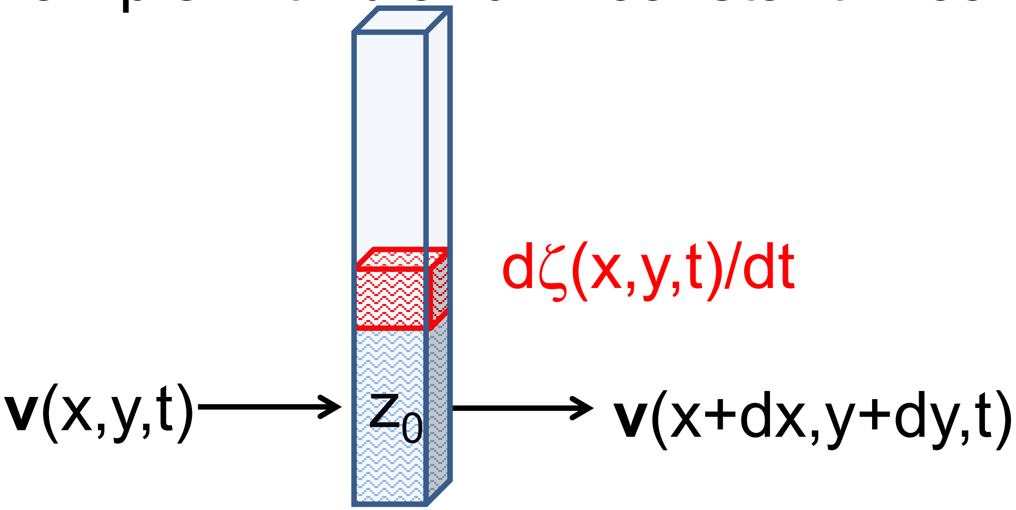
$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

Special case, where *b* and *h* are constant --

For constant *b* and *h*:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\left(v(x,t)\right)$$

Example with *b* and *h* constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla \zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2 \zeta = 0$

For uniform channel:

Surface wave equation:

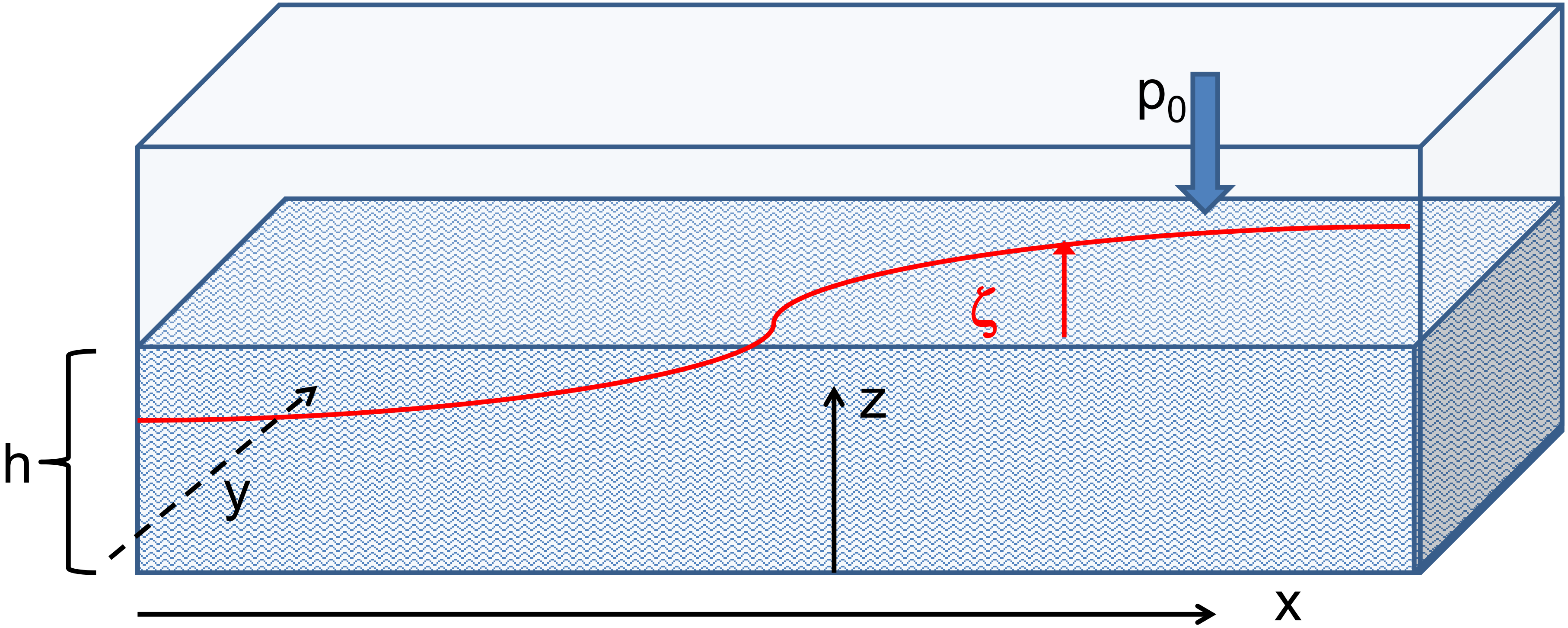
$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \qquad\qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

More details: -- recall setup --

Consider a container of water with average height h and surface h+$\zeta(x,y,t)$



Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

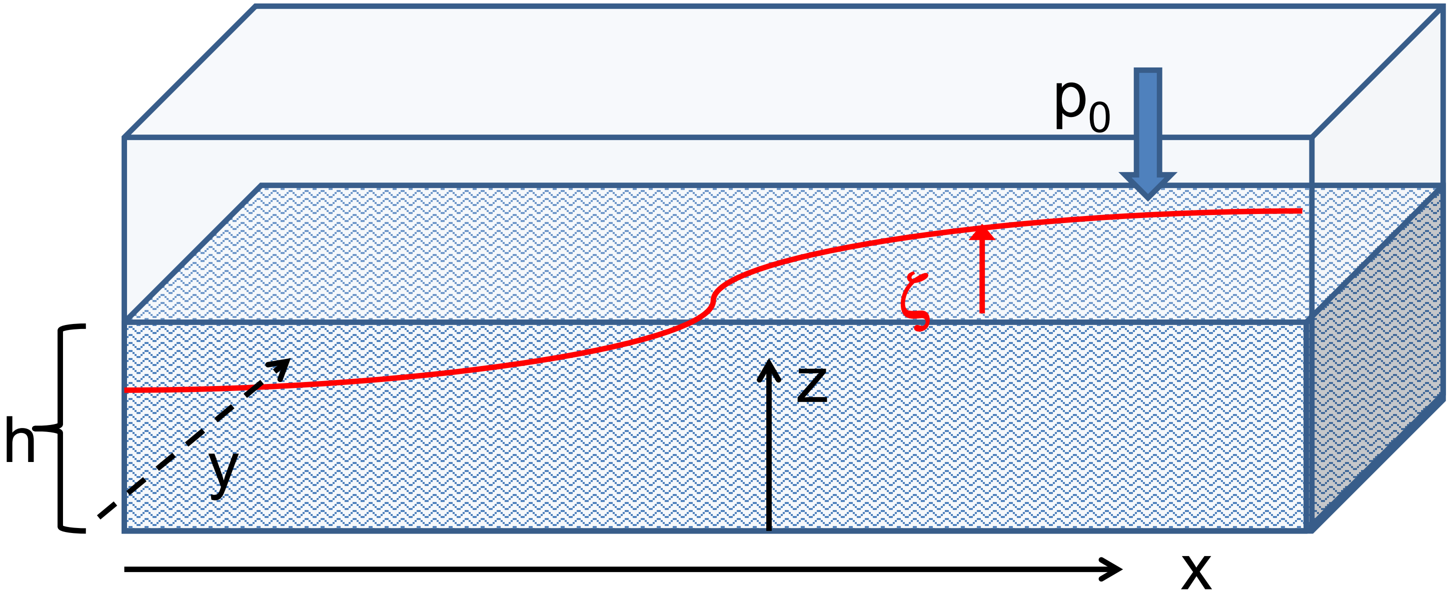
Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant}$$

(We have absorbed p$_0$ in "constant")

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y}$$

where $v_{x,y} = v_{x,y}\left(x, y, h + \zeta, t\right)$

Full equations:

Within fluid: $0 \leq z \leq h+\zeta$

$$-\frac{\partial \Phi}{\partial t}+\tfrac{1}{2} v^2+g(z-h)=\text{constant}$$

(We have absorbed p_0 in “constant”)

$$-\nabla^2 \Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta(x, y, t)$

$$\frac{d \zeta}{d t}=\frac{\partial \zeta}{\partial t}+v_x \frac{\partial \zeta}{\partial x}+v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y}=v_{x,y}(x, y, h+\zeta, t)$$

Linearized equations:

For $0 \leq z \leq h+\zeta$: $-\frac{\partial \Phi}{\partial t}+g(z-h)=0 \qquad -\nabla^2 \Phi=0$

At surface: $z=h+\zeta \qquad \frac{d \zeta}{d t}=\frac{\partial \zeta}{\partial t}=v_z(x, y, h+\zeta, t)$

$$-\frac{\partial \Phi(x, y, h+\zeta, t)}{\partial t}+g \zeta=0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \leq z \leq h+\zeta$: $\nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t)=0$

Consider and periodic waveform: $\Phi(x,z,t)=Z(z)\cos\left(k(x-ct)\right)$

$$\Rightarrow \left(\frac{d^2}{dz^2}-k^2\right)Z(z)=0$$

Boundary condition at bottom of tank: $v_z(x,0,t)=0$

$$\Rightarrow \frac{dZ}{dz}\left(0\right)=0 \qquad Z(z)=A\cosh(kz)$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface:} \quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z}$$

$$-\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z} = 0$$

$$\text{For} \qquad \Phi(x,\left(h+\zeta\right),t) = A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)$$

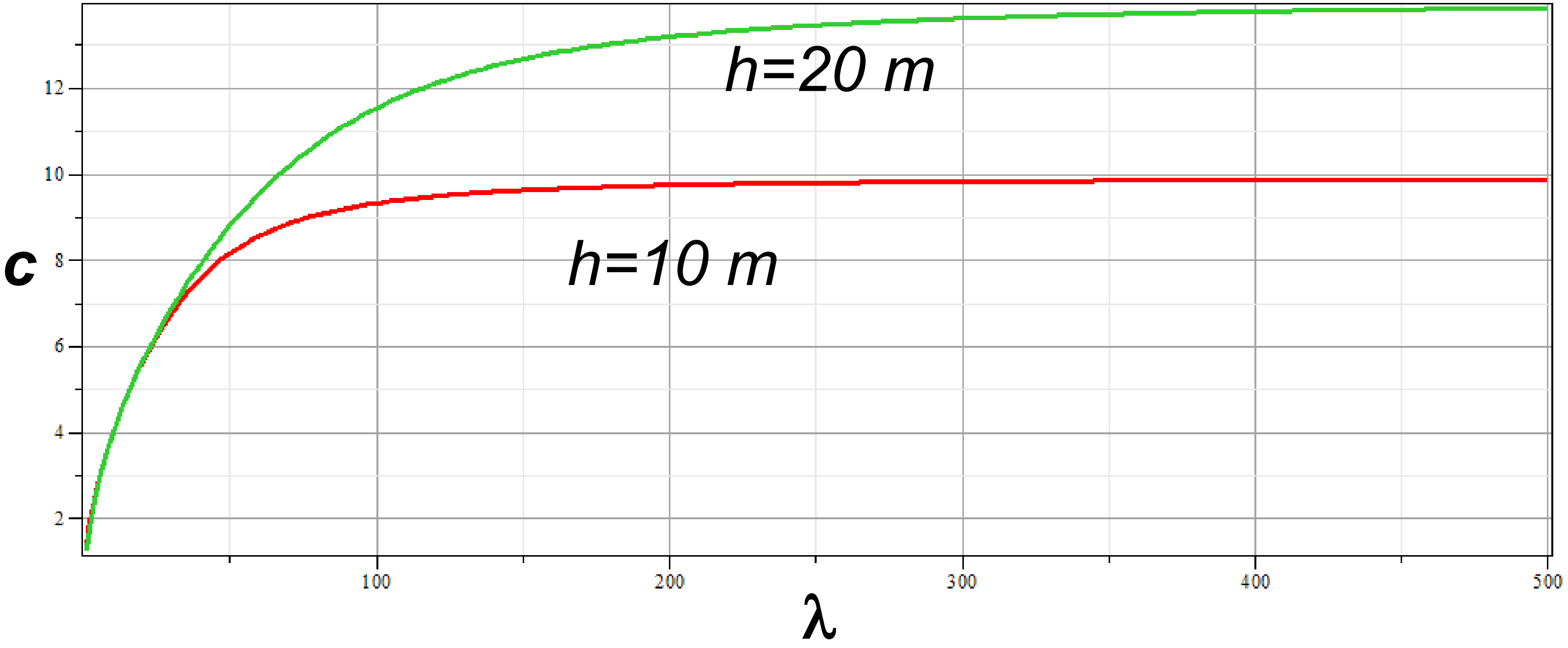
$$A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}\right) = 0$$

$$\Rightarrow \mathrm{c}^2 = \frac{g}{k}\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

$$\text{Assuming } \zeta << h: \qquad c^2 = \frac{g}{k}\tanh(kh) \qquad \lambda = \frac{2\pi}{k}$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h, \; c^2 \approx gh$$

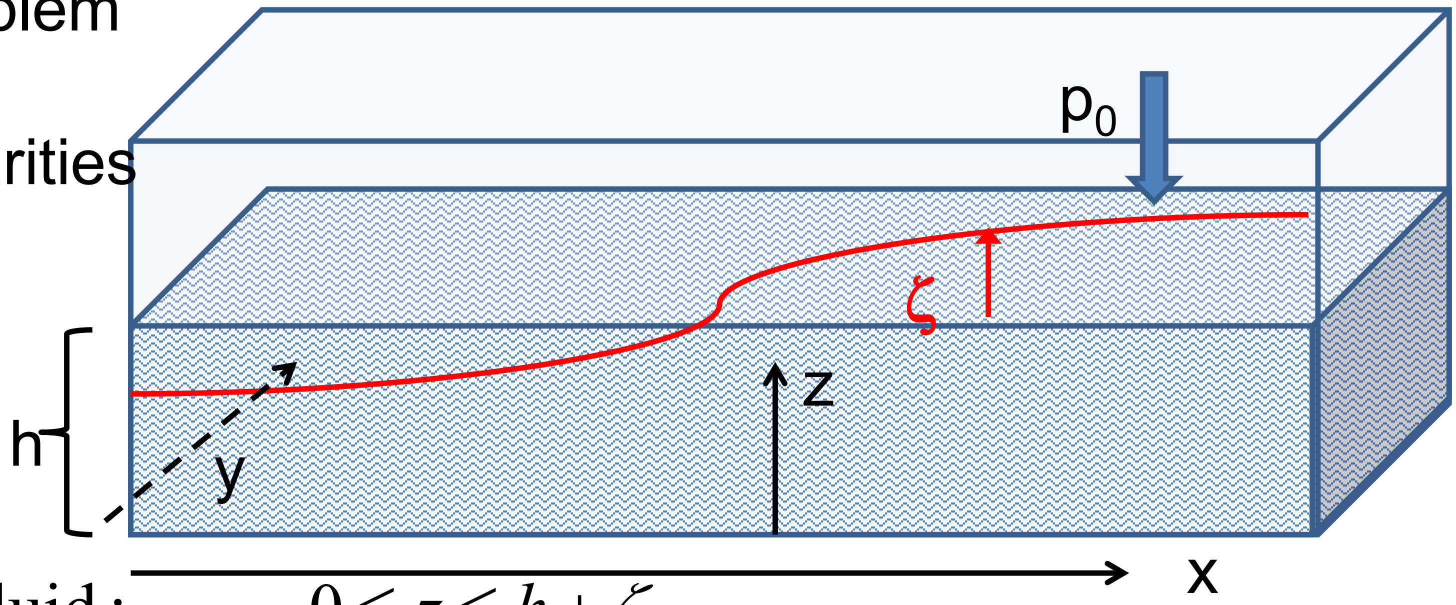
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h, \quad c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem including non-linearities



Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}v^2+g(z-h)=\text{constant}$$

(We have absorbed p_0 in our constant.)

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta(x,y,t)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}$$

where $v_{x,y}=v_{x,y}(x,y,h+\zeta,t)$