PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Plan for Lecture 34:

Chapter 10 in F & W: Surface waves

1. Water waves in a channel
2. Wave-like solutions; wave speed

In today's lecture we will investigate transverse waves at the surface of a channel of water.

WAKE FOREST UNIVERSITY

Department of Physics

Colloquium

Thursday, Nov. 12, 2020
4 PM

Britt Lundgren, PhD
Assistant Professor
Department of Physics and Astronomy
UNC Asheville
Asheville, NC

"Shedding Light on the Gaseous Halos of Galaxies in the Early Universe"

Simulations predict that galaxy evolution is regulated by the accretion, expulsion, cooling, and heating of gas in the halos that surround galaxies out to hundreds of kiloparsecs. Distant quasars can be used to backlight and detect circumgalactic gas in absorption, and to date, tens of thousands of intervening absorbers have been detected in large spectroscopic quasar surveys.

Thursday's colloquium speaker is a Professor of Physics and Astronomy at UNC Asheville who will be talking about simulations of astronomical observables to better understand galaxies for example.

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Update to schedule including a homework dealing with today’s topic.

PHY 711 -- Assignment #22

Nov. 11, 2020

Start reading Chapter 10 in **Fetter & Walecka**.

1. Work Problem 10.3 at the end of Chapter 10 in **Fetter and Walecka**.

Homework problem.

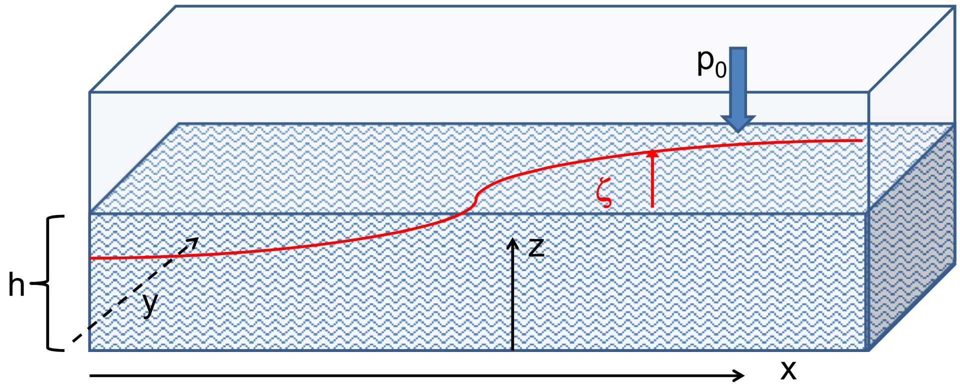
Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)

Atmospheric pressure is in equilibrium with the surface of water

$p_0 = \rho g\left(h+\zeta\right)$ Here ρ represents density of water



Defining the system and the notation.

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\left(\zeta(x,y,t) + h - z\right)$$ within the water

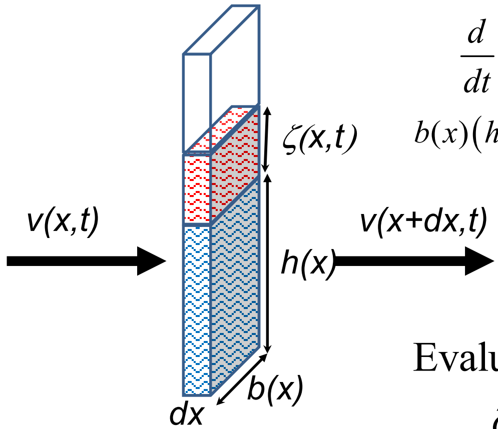
Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

Hydrodynamic equations for this case.

Consider a surface $\zeta(x,t)$ wave moving in the x-direction in a channel of width $b(x)$ and height $h(x)$:



Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

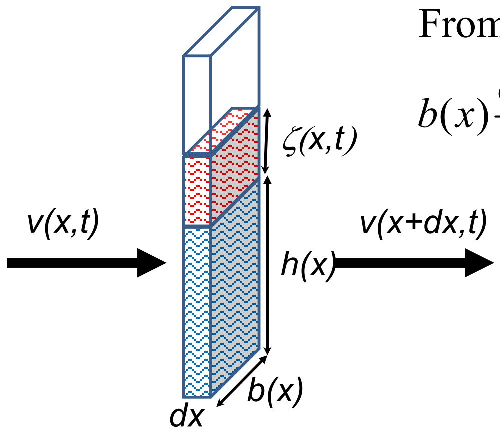
dV: $b(x)\big(h(x)+\zeta(x,t)\big)dx$

$d\mathbf{A}$: $b(x)\big(h(x)+\zeta(x,t)\big)\hat{\mathbf{x}}$

Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Considering an increment along the propagation direction including the effects of the continuity equation.



From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big((\kappa x)b_0 v(x,t)\big)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

Some details for the homework problem which is a special case.

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x} \right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t} \right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2} \right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x} \right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1 \right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} \right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} \right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$

More details pertaining to the homework problem.

Example continued

$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left(\frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

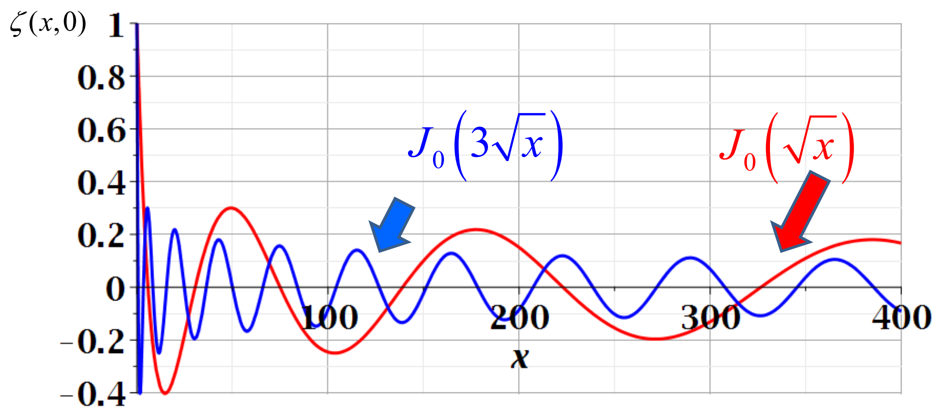
$$\Rightarrow \zeta(x,t) = C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

Check:

$$-\omega^2 C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left(\frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

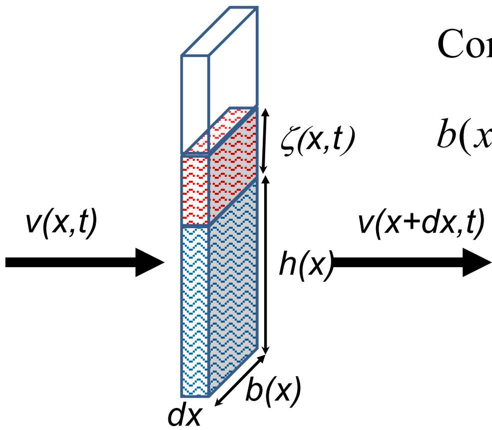
Continued.

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$



Continued.

Another example:



Continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

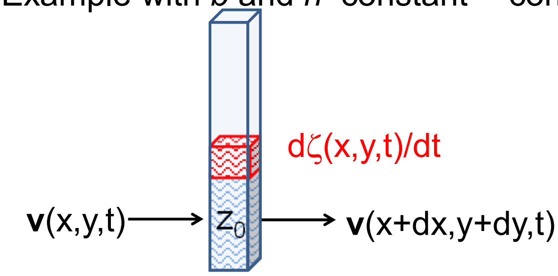
Special case, where *b* and *h* are constant --

For constant b and h:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\big(v(x,t)\big)$$

A simpler example.

Example with *b* and *h* constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2\zeta = 0$

Considering the surface height.

For uniform channel:

Surface wave equation:

$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \qquad\qquad c^2 = gh$$

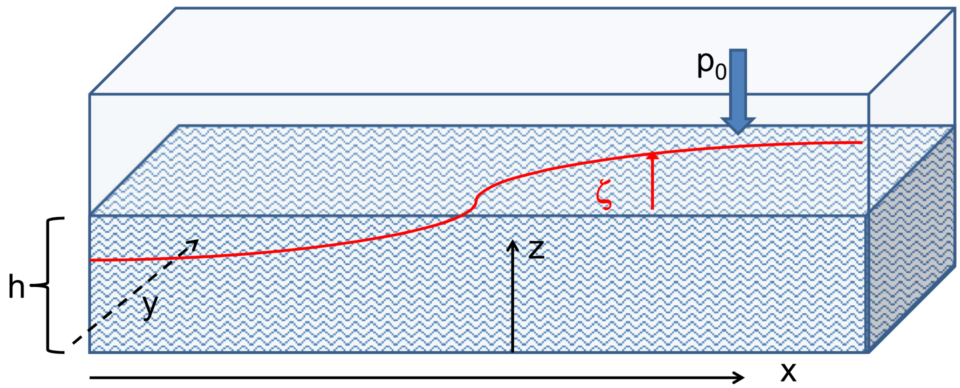
More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

For the simple case, we find the wave equation for the surface height. In the following slides, we will find a more complete solution depends on the wavelength the of surface wave.

More details: -- recall setup --

Consider a container of water with average height h and surface h+ζ(x,y,t)



Some details for the more general case.

Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

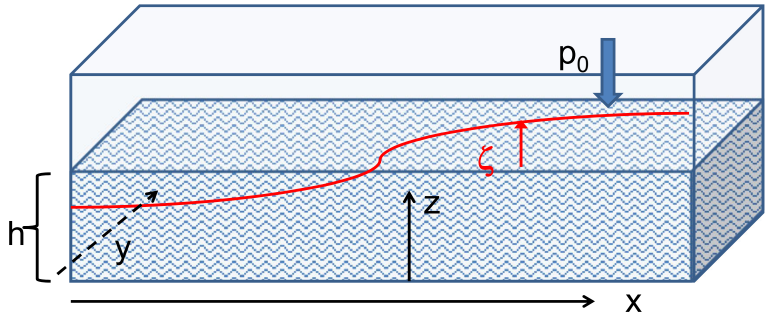
$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$

Considering the case of irrotational flow.



Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g\left(z-h\right) = \text{constant}$$

(We have absorbed p_0 in "constant")

$$-\nabla^2 \Phi = 0$$

At surface: $z = h+\zeta$ with $\zeta = \zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y}$$

where $v_{x,y} = v_{x,y}\left(x,y,h+\zeta,t\right)$

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Considering the equations within the wave and at the surface.

Full equations:

Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}v^2+g(z-h)=\text{constant}$$

(We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta(x,y,t)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y}=v_{x,y}(x,y,h+\zeta,t)$$

Linearized equations:

For $0 \le z \le h+\zeta$: $-\frac{\partial \Phi}{\partial t}+g(z-h)=0 \qquad -\nabla^2\Phi=0$

At surface: $z=h+\zeta \qquad \frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}=v_z(x,y,h+\zeta,t)$

$$-\frac{\partial\Phi(x,y,h+\zeta,t)}{\partial t}+g\zeta=0$$

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Taking the linear limit.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \le z \le h + \zeta$: $\nabla^2 \Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x,z,t) = 0$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad Z(z) = A\cosh(kz)$$

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Solution for the linear equations.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface}: \quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z}$$

$$-\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z} = 0$$

$$\text{For} \qquad \Phi(x, (h+\zeta), t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

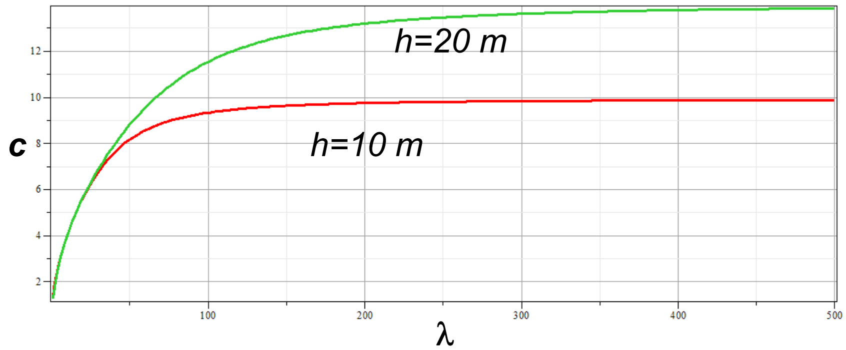
$$\Rightarrow \mathrm{c}^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

An expression for c.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

Assuming $\zeta << h$: $\quad c^2 = \frac{g}{k}\tanh(kh) \qquad \lambda = \frac{2\pi}{k}$



Evaluating c as a function of wavelength.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

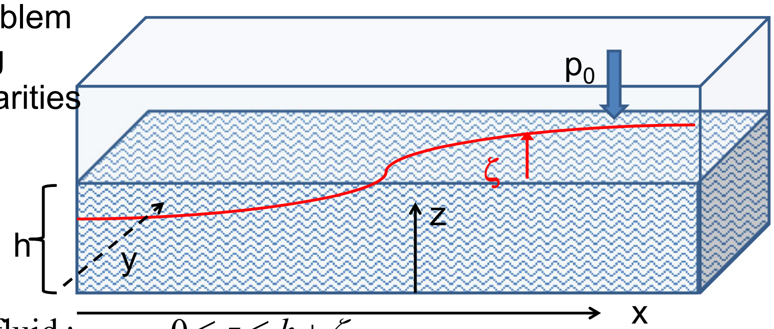
$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h$, $c^2 \approx gh$

(solutions are consistent with previous analysis)

Form of the surface wave form.

General problem
including
non-linearities



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2 \Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Introducing the equations beyond the linear approximation that we will cover next time.