PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Plan for Lecture 35: Chapter 10 in F & W

Surface waves

- **Summary of linear surface wave solutions**
- **Non-linear contributions and soliton solutions**

This material is covered in Chapter 10 of your textbook using similar notation.

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule for weekly one-on-one meetings (EST)

Nick – 11 AM Monday
Tim – 9 AM Tuesday
Gao – 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Your questions –

From Gao –

1. In what situation, the velocity potential phi in Bernoulli's equation satisfies d(phi)/d(t)=0？
2. In visualization, what do the red line and the blue line stand for?
3. Why does zero vertical velocity at the bottom of a pool ensure all odd derivatives vanish from the Taylor expansion?

From Nick –

I have a conceptual question. My misunderstanding is probably mostly about the setup and remembering what everything means. But I was wondering why is this true?

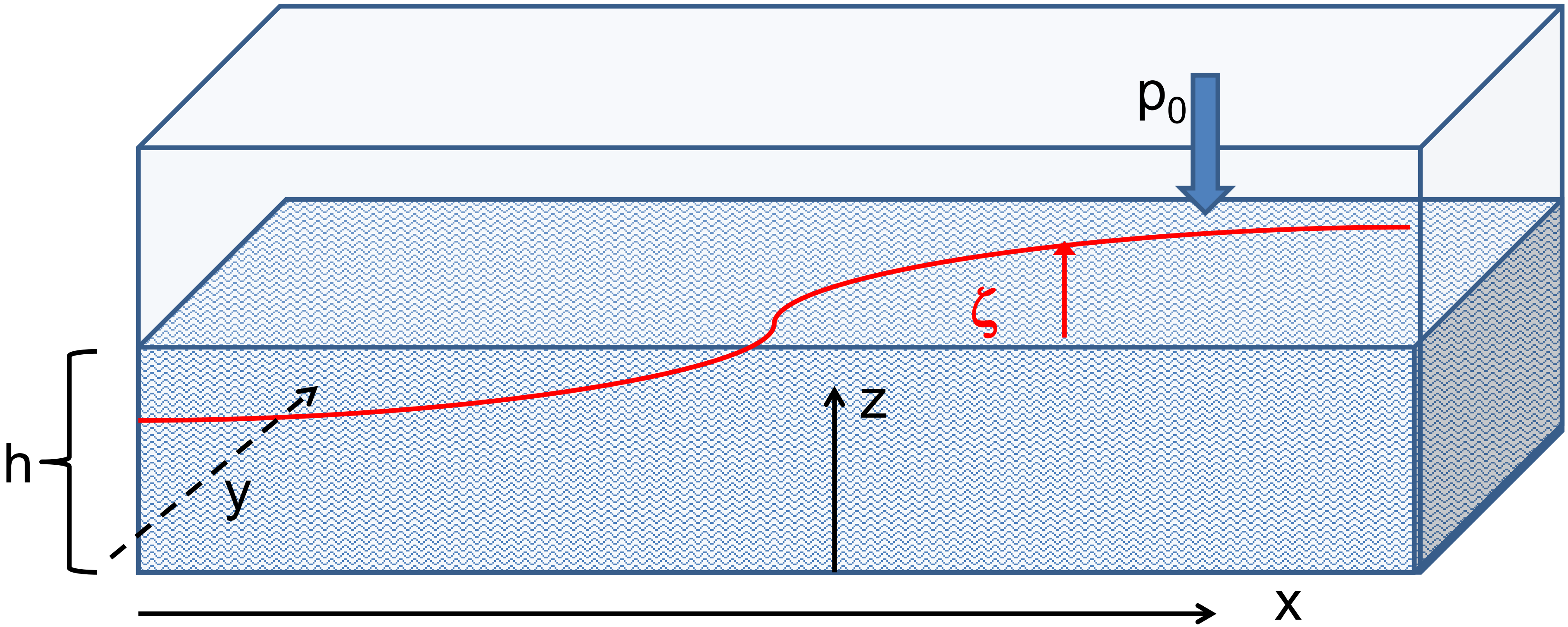
$$\text{At surface:} \quad z = h + \zeta \qquad \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} = v_z(x, y, h+\zeta, t)$$

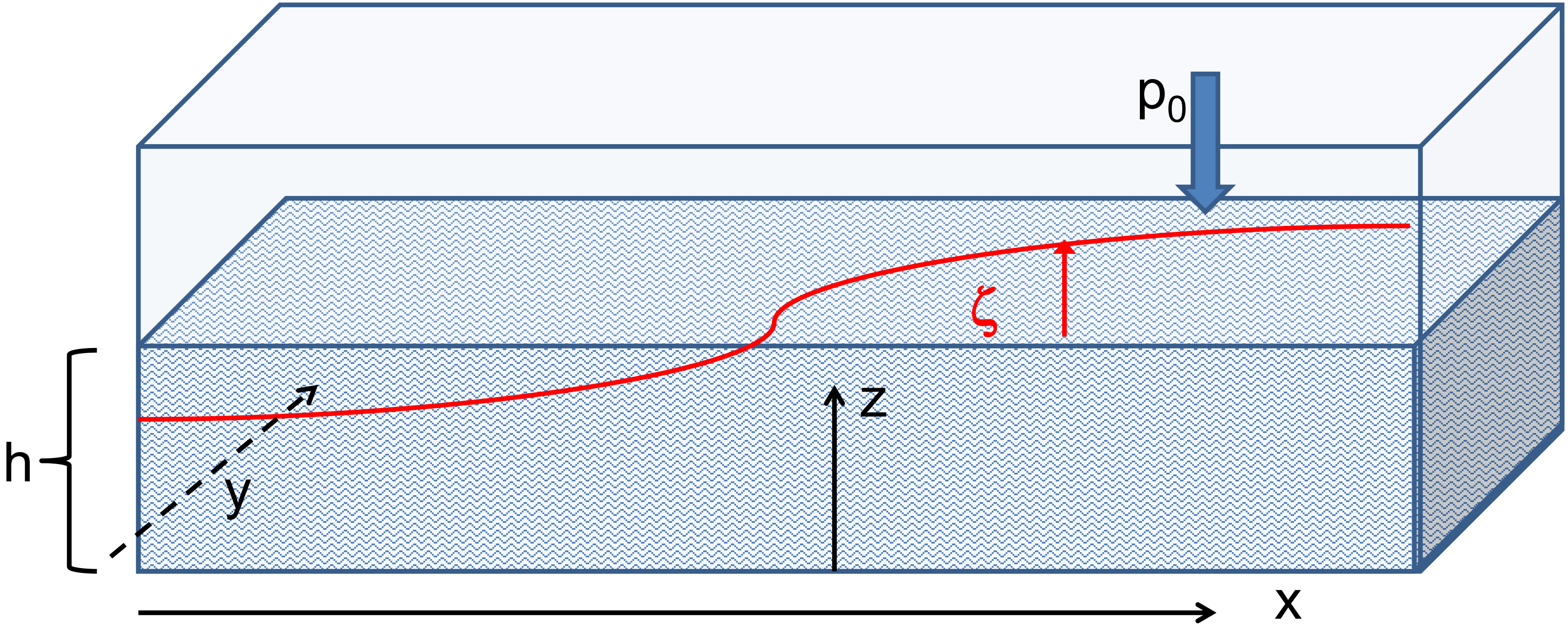
In particular, why is

$$\frac{\partial \zeta}{\partial t} = v_z(x, y, h+\zeta, t)$$

Consider a container of water with average height h and surface $h+\zeta(x,y,t)$

Atmospheric pressure p_0 is in equilibrium at the surface





Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Continuity equation within the fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0$$

For irrotational flow -- $\mathbf{v} = -\nabla \Phi$

Linearized equation: $\nabla \left(-\frac{\partial \Phi}{\partial t} + g(z-h) + \frac{p}{\rho} \right) = 0$

At surface: $z = h + \zeta \qquad -\frac{\partial \Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

Keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \le z \le h+\zeta$: $\nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t)=0$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2}-k^2\right)Z(z)=0$$

Boundary condition at bottom of tank: $v_z(x,0,t)=0$

$$\Rightarrow \frac{dZ}{dz}(0)=0 \qquad Z(z)=A\cosh(kz)$$

At surface: $z=h+\zeta$ $\quad \frac{\partial\zeta}{\partial t} = v_z\left(x,h+\zeta,t\right) = -\frac{\partial\Phi\left(x,h+\zeta,t\right)}{\partial z}$

Also: $$-\frac{\partial\Phi\left(x,h+\zeta,t\right)}{\partial t}+g\zeta+\frac{p_0}{\rho}=0$$

$$\Rightarrow -\frac{\partial^2\Phi\left(x,h+\zeta,t\right)}{\partial t^2}+g\frac{\partial\zeta}{\partial t} = -\frac{\partial^2\Phi\left(x,h+\zeta,t\right)}{\partial t^2}-g\frac{\partial\Phi\left(x,h+\zeta,t\right)}{\partial z}=0$$

Velocity potential: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

At surface: $\Phi(x,\left(h+\zeta\right),t) = A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)$

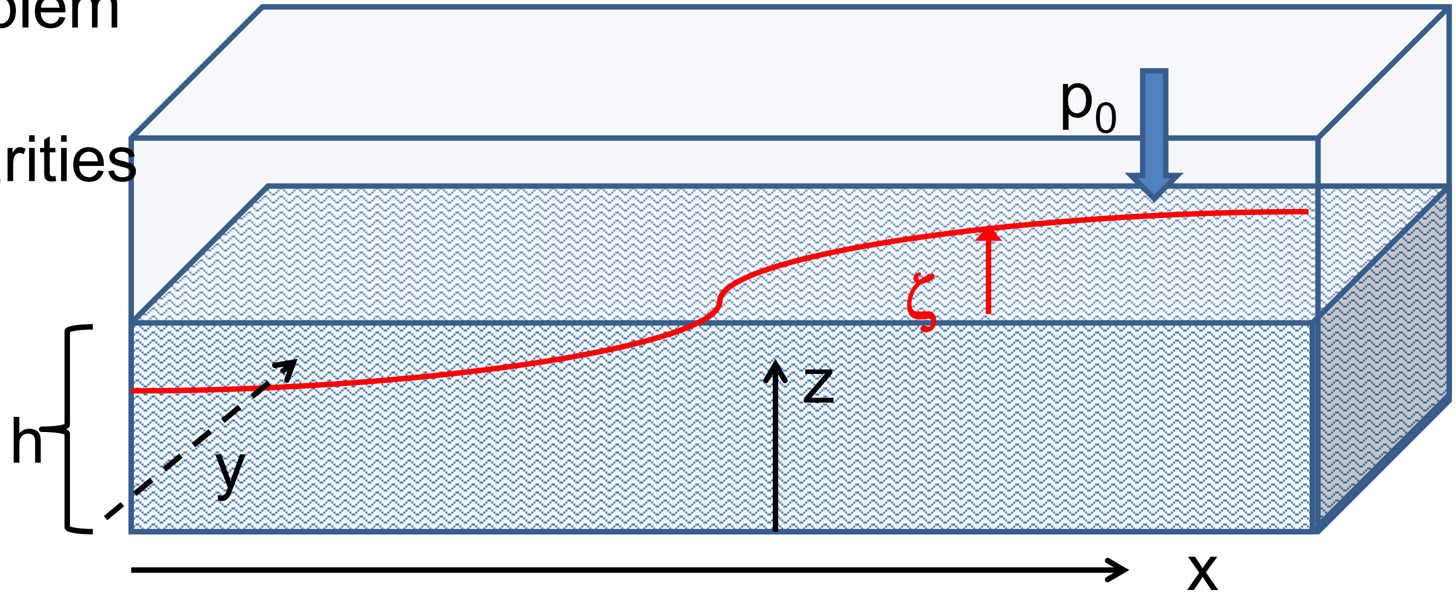
$$A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}\right) = 0$$

$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))} \approx \frac{g}{k}\tanh(kh)$$

Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors *k*. Additionally, your text considers the effects of surface tension. In this lecture, we will focus on the effects of the non-linear effects of Euler and continuity equations.

Surface waves in an incompressible fluid

General problem including non-linearities



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g\left(z - h\right) = \text{constant} \qquad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x, y, z, t)}{\partial z}\right|_{z = h + \zeta}$$

where $v_{x,y} = v_{x,y}\left(x, y, h + \zeta, t\right)$

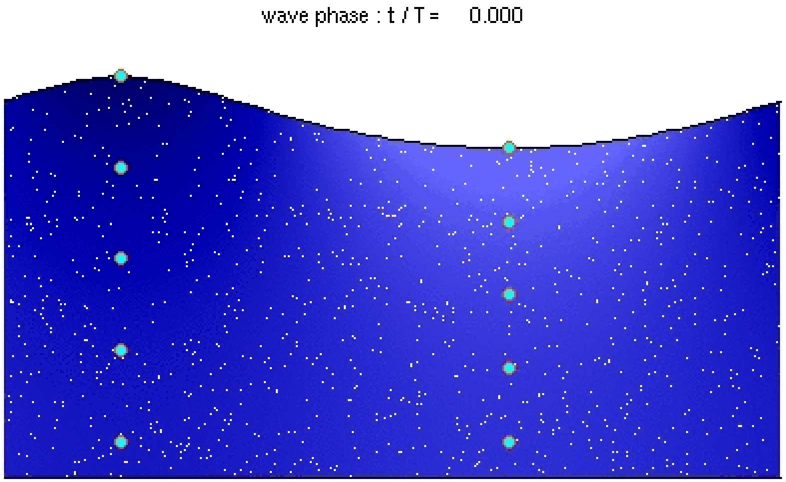
Your question – why is the following true?

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x, y, z, t)}{\partial z}\right|_{z=h+\zeta}$$

where $v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$

Note that $v_z(x, y, h+\zeta, t) = \frac{d\zeta}{dt}$



From wikipedia

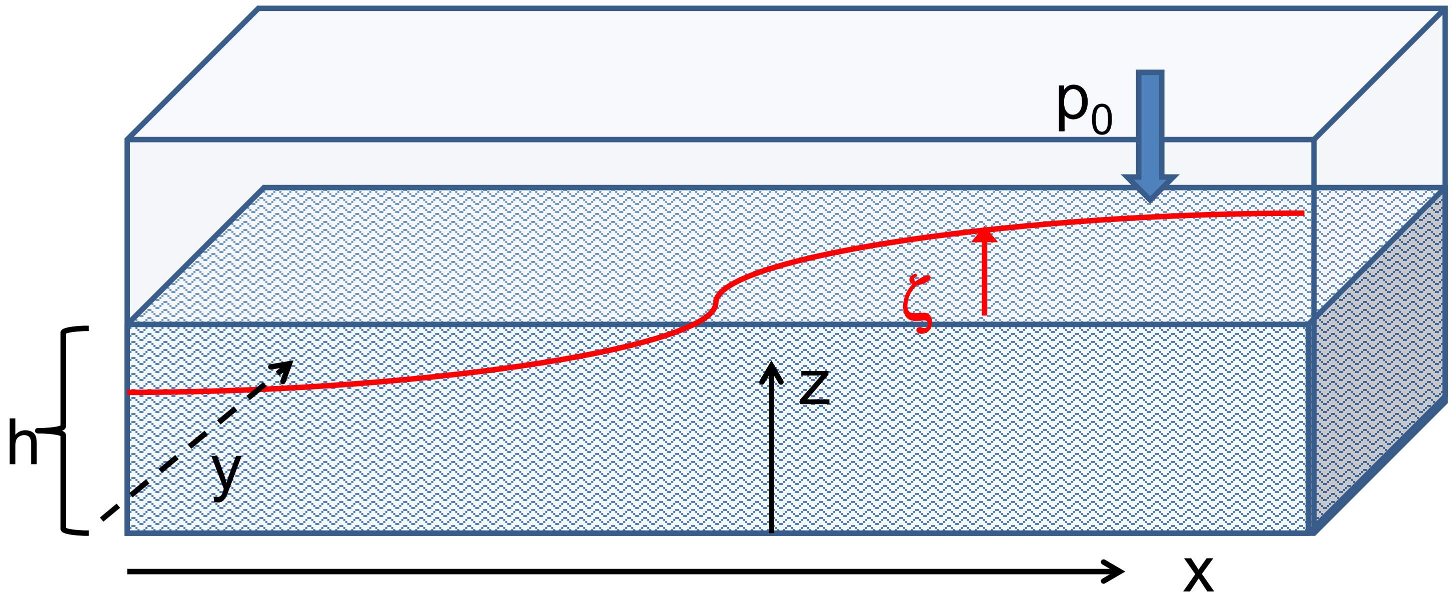
Your question -- In what situation, the velocity potential phi in Bernoulli's equation satisfies d(phi)/d(t)=0?

Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant} \qquad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

One example of $\frac{\partial \Phi}{\partial t} \approx 0$ would be for $v^2 \approx 0$ and $z = h$.



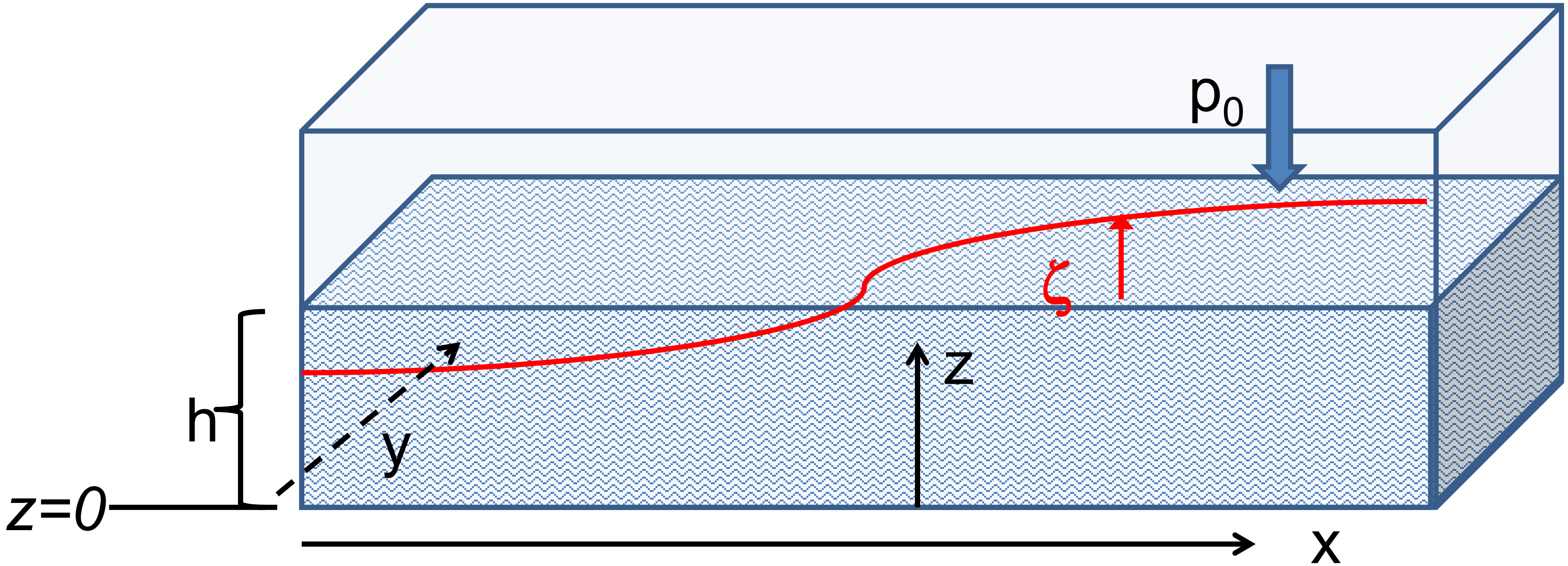
Further simplifications; assume trivial y - dependence

$$\Phi = \Phi(x, z, t) \qquad \zeta = \zeta(x, t)$$

Within fluid: $\quad 0 \le z \le h + \zeta$

At surface: $\quad v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \,\mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$

Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \leq z \leq h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2}_{v_z^2}\right] + g(z-h) = 0.$$

Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\Big|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v}\cdot\nabla\zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$

Analysis assuming water height *z* is small relative to variations in the direction of wave motion *(x)*

Taylor's expansion about $z = 0$:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom suggest that to a good approximation, that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to *z* to derivatives with respect to *x*.

$$\Phi(x,z,t) \approx \Phi(x,0,t) + \cancel{z\frac{\partial \Phi}{\partial z}(x,0,t)} + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \cancel{\frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t)} + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

$$\Rightarrow \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots$

Question -- Why does zero vertical velocity at the bottom of a pool ensure all odd derivatives vanish from the Taylor expansion?

Comment – You are right it is not a rigorous, but an approximate result.

Example from linear case: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

In this case all odd derivatives $\left.\frac{\partial^n \Phi(x,z,t)}{\partial z^n}\right|_{z=0} = 0$

One can think of other counter examples of functions for which the first derivative vanishes but the third derivative does not.

Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x,h,t)}{\partial z} \approx h\frac{\partial^2 \Phi(x,0,t)}{\partial x^2} = -\frac{\partial \zeta(x,t)}{\partial t} \qquad -\frac{\partial \Phi(x,h,t)}{\partial t} \approx -\frac{\partial \Phi(x,0,t)}{\partial t} = -g\zeta(x,t)$$

Decoupled equations: $$\frac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2 \Phi(x,0,t)}{\partial x^2}.$$

➔linear wave equation with $c^2 = gh$

Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial \Phi(x,z,t)}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2+\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2\right]\Bigg|_{z=h+\zeta} + g\zeta(x,t)=0.$$

Consistency of surface velocity

$$-\frac{\partial \Phi(x,z,t)}{\partial z}+\frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x}-\frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta}=0$$

Representation of velocity potential from Taylor′s expansion:

$$\Phi(x,z,t)\approx \Phi(x,0,t)-\frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t)+\frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots$$

Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\Rightarrow \frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

These equations represent non-linear coupling of $\phi(x,t)$ and $\zeta(x,t)$.

Coupled equations: $-\frac{\partial\phi}{\partial t}+\frac{h^2}{2}\frac{\partial^3\phi}{\partial t\partial x^2}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+g\zeta=0.$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial\phi}{\partial x}\right)-\frac{h^3}{3!}\frac{\partial^4\phi}{\partial x^4}-\frac{\partial\zeta}{\partial t}=0.$$

Traveling wave solutions with new notation:

$u\equiv x-ct \qquad \phi(x,t)\equiv\chi(u) \qquad$ and $\quad \zeta(x,t)\equiv\eta(u)$

Note that the wave "speed" c will be consistently determined

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\chi'=-\frac{g}{c}\eta+\frac{h^2}{2}\chi'''-\frac{1}{2c}(\chi')^2\approx-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow(h+\eta)\frac{d\chi(u)}{du}-\frac{h^3}{6}\frac{d^3\chi(u)}{du^3}+c\eta(u)=0$$

Now we can express $\frac{d\chi(u)}{du}=\chi'$ in terms of η:

$$\chi'\approx-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3 g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

$$\text{Note:}\quad c^2=gh+\ldots$$

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

$$\Rightarrow \left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)\quad \text{where } \eta_0 \text{ is a constant}$$

Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}$ $\qquad\Rightarrow\frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$

Multiply equation by $\eta'(u)$ $\qquad\Rightarrow\frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

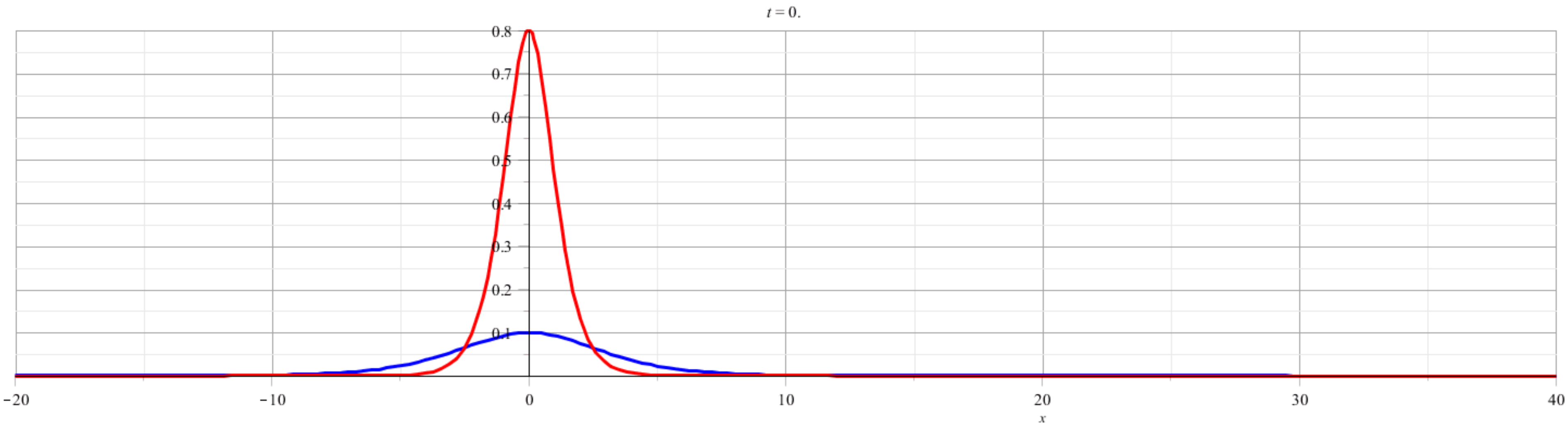
Integrate wrt u and assume solution vanishes for $u\to\infty$

$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

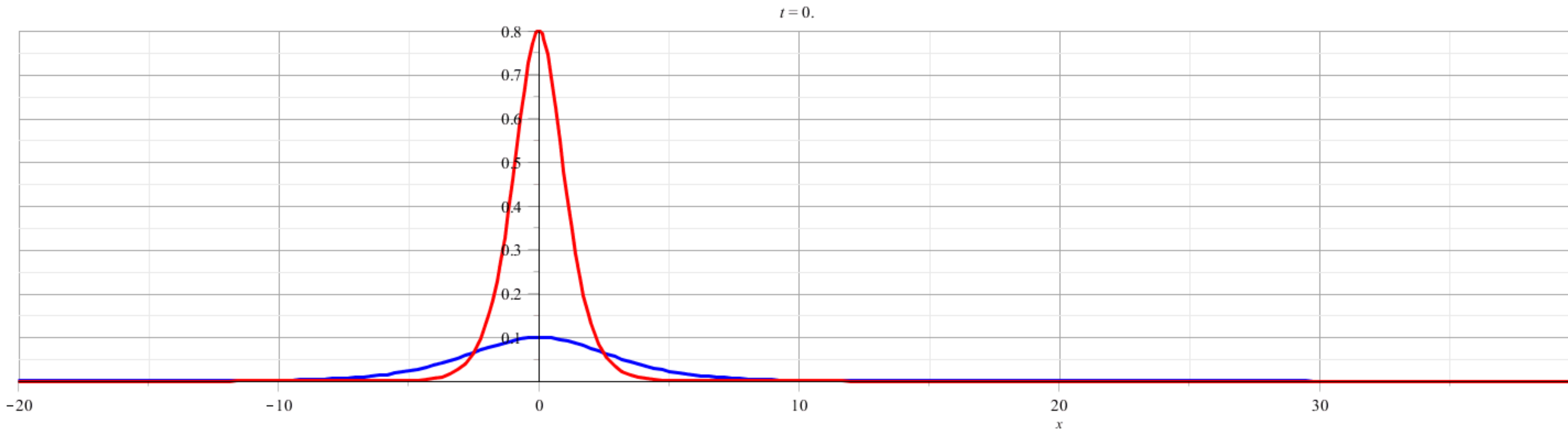
$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du \qquad\Rightarrow\eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$



Your question -- In visualization, what do the red line and the blue line stand for?

Two soliton solutions with different amplitudes



Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right].$$

More details

Modified surface amplitude equation in terms of η :

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Some identities: $\frac{\eta_0}{h}=1-\frac{gh}{c^2};\quad \frac{\partial\eta}{\partial t}=-c\frac{d\eta}{du};\quad \frac{\partial\eta}{\partial x}=\frac{d\eta}{du}.$

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta'-\frac{h^2}{3}\eta'''-\frac{3}{h}\eta\eta'=0.$$

Expression in terms of x and t:

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t}-\frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3}-\frac{3}{h}\eta\frac{\partial\eta}{\partial x}=0.$$

Expression in terms of $\overline{x}$ and $\overline{t}$:

$$\frac{\partial\eta}{\partial\overline{t}}+6\eta\frac{\partial\eta}{\partial\overline{x}}+\frac{\partial^3\eta}{\partial\overline{x}^3}=0.$$

Summary

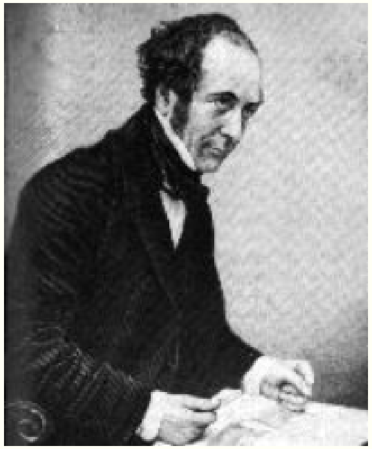
Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

Photo of canal soliton http://www.ma.hw.ac.uk/solitons/

John Scott Russell and the solitary wave



Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

https://www.macs.hw.ac.uk/~chris/scott_russell.html

``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
(Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.