PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Discussion for Lecture 36: Chap. 11 in F&W

Heat conduction

1. Basic equations
2. Boundary value problems

29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule for weekly one-on-one meetings (EST)

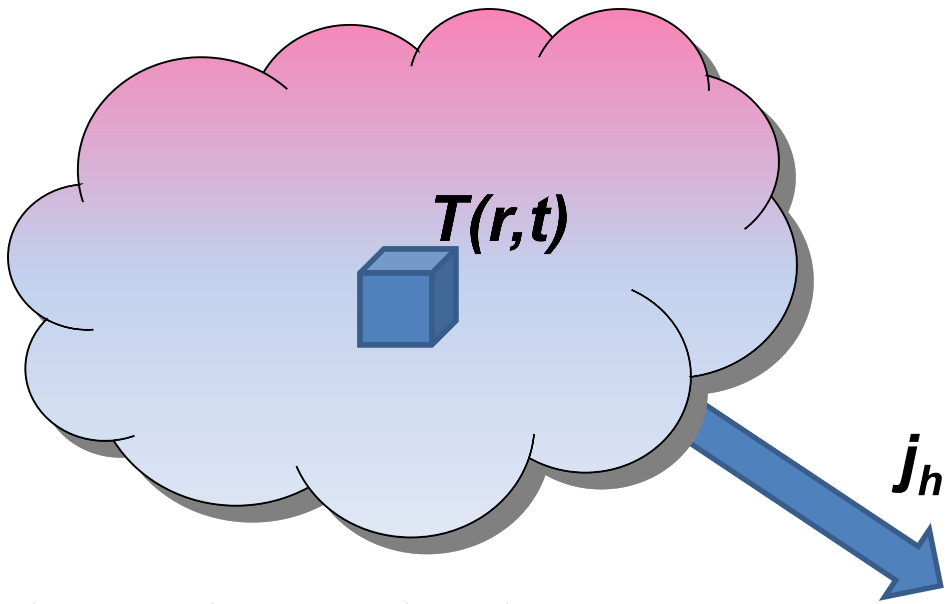
Nick – 11 AM Monday
Tim – 9 AM Tuesday – Possibly Wed. at 11 AM?
Gao – 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Your questions –
From Nick –
1. Can you explain the erf function. I've never really understood it. I'm not sure I've actually ever had someone explain it. I've just seen it appear in places before.

From Gao –
1. In the real world, what system has a temperature distribution such as discussed for the standing wave solutions.
2. Does this expression say the temperature transmits along the Z axis?

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure *p*

Notation: Heat added to system -- $dQ = TdS$

External work done on system -- $dW = -pdV$

Internal energy -- $dE = dQ + dW = TdS - pdV$

Entropy -- dS

Enthalpy -- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \int \rho c_p d^3 r$$

More generally, note that c_p can depend on *T*; we are assuming that dependence to be trivial.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T\left(\mathbf{r},t\right) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T\left(\mathbf{r},t\right)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T\left(\mathbf{r},t\right)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

$$\text{Empirically:} \quad \mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

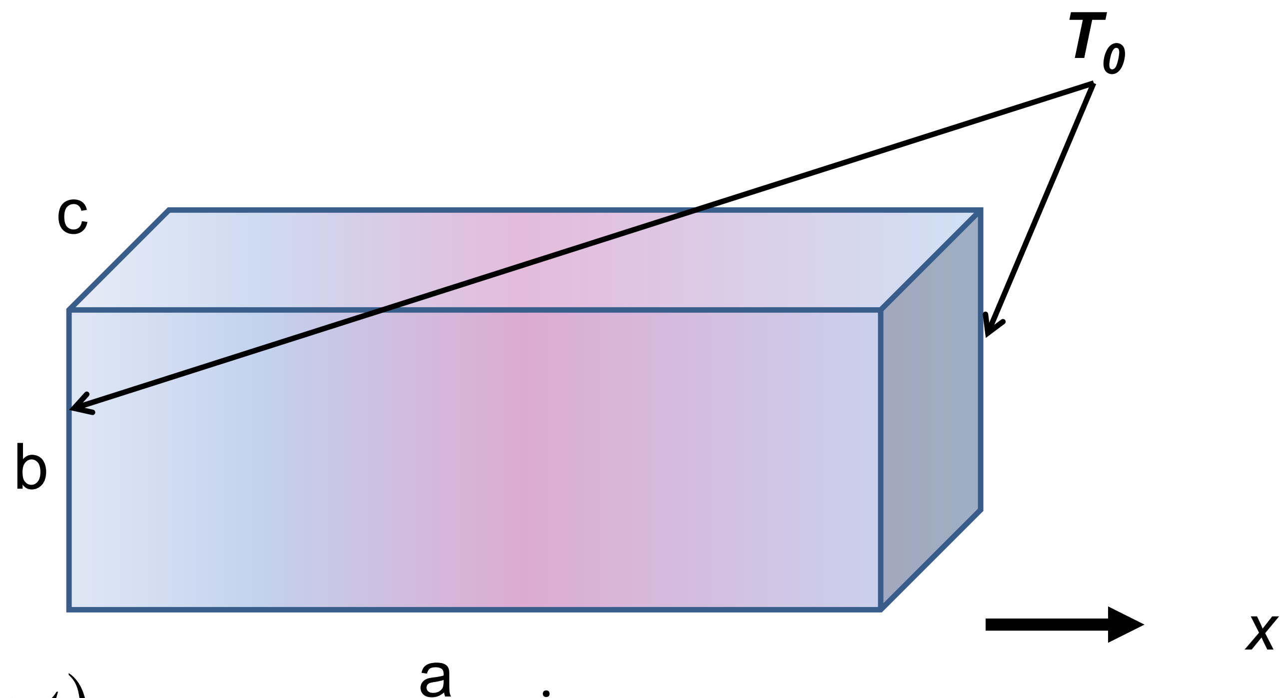
$$\kappa \equiv \frac{k_{th}}{\rho c_p}$$ thermal diffusivity

https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm

Typical values (m^2/s)

Air	2×10^{-5}
Water	1×10^{-7}
Copper	1×10^{-4}

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$

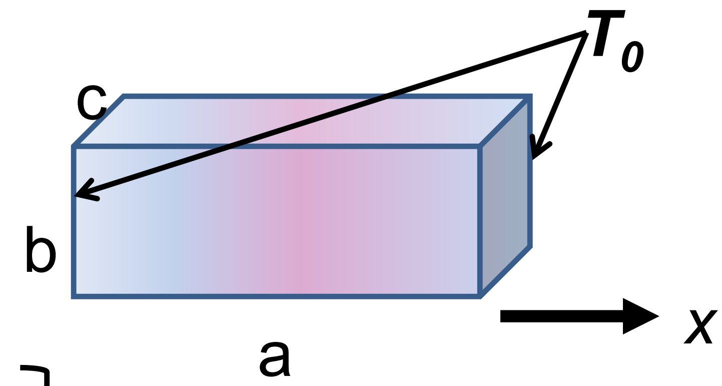
Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Have you ever encountered the following equation in other contexts and if so where?

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

Boundary value problems for heat conduction

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$



$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

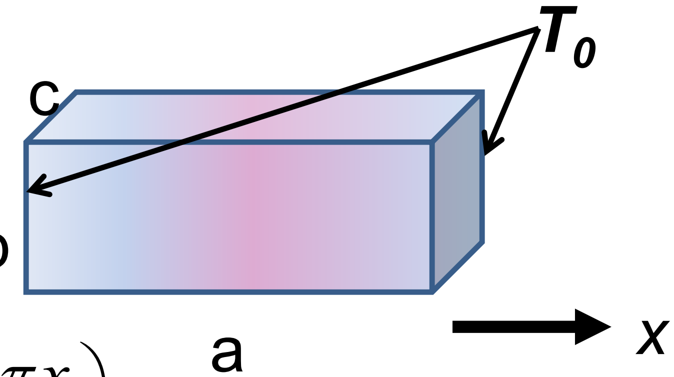
Assuming thermally insulated boundaries

Separation of varables: $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2 X}{dx^2} = -\alpha^2 X \quad \frac{d^2 Y}{dy^2} = -\beta^2 Y \quad \frac{d^2 Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa\left(\alpha^2 + \beta^2 + \gamma^2\right) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

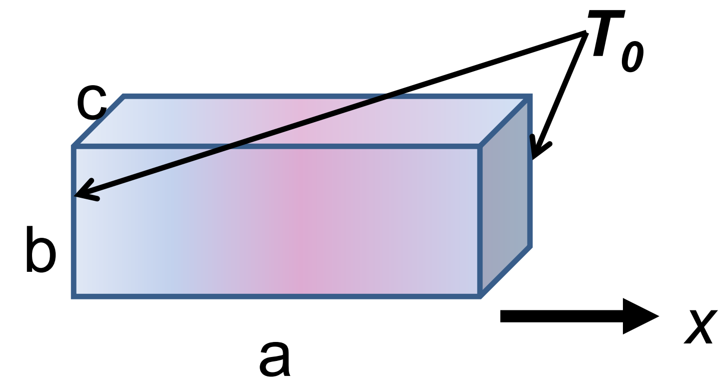
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



Full solution:

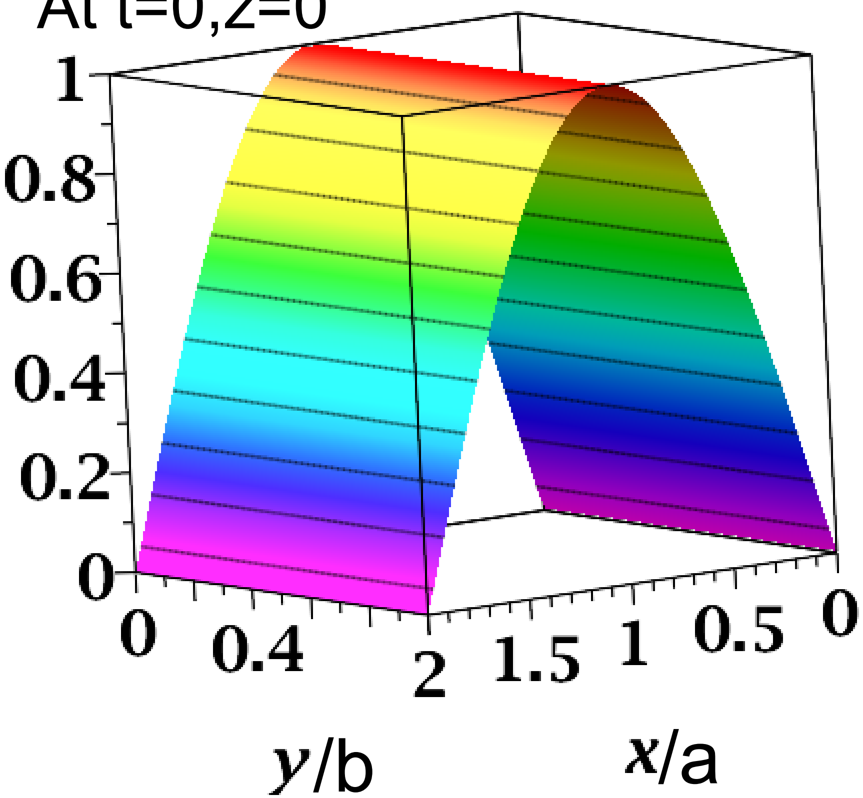
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right) e^{-\lambda_{mnp} t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

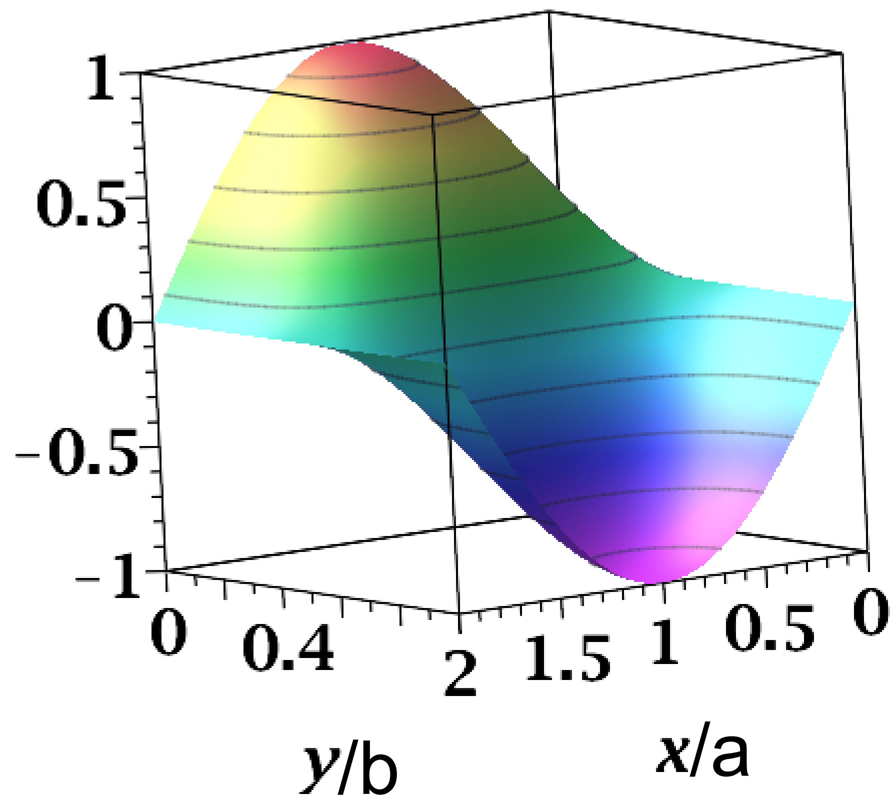
Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

At t=0,z=0



$m=1, n=0, p=0$



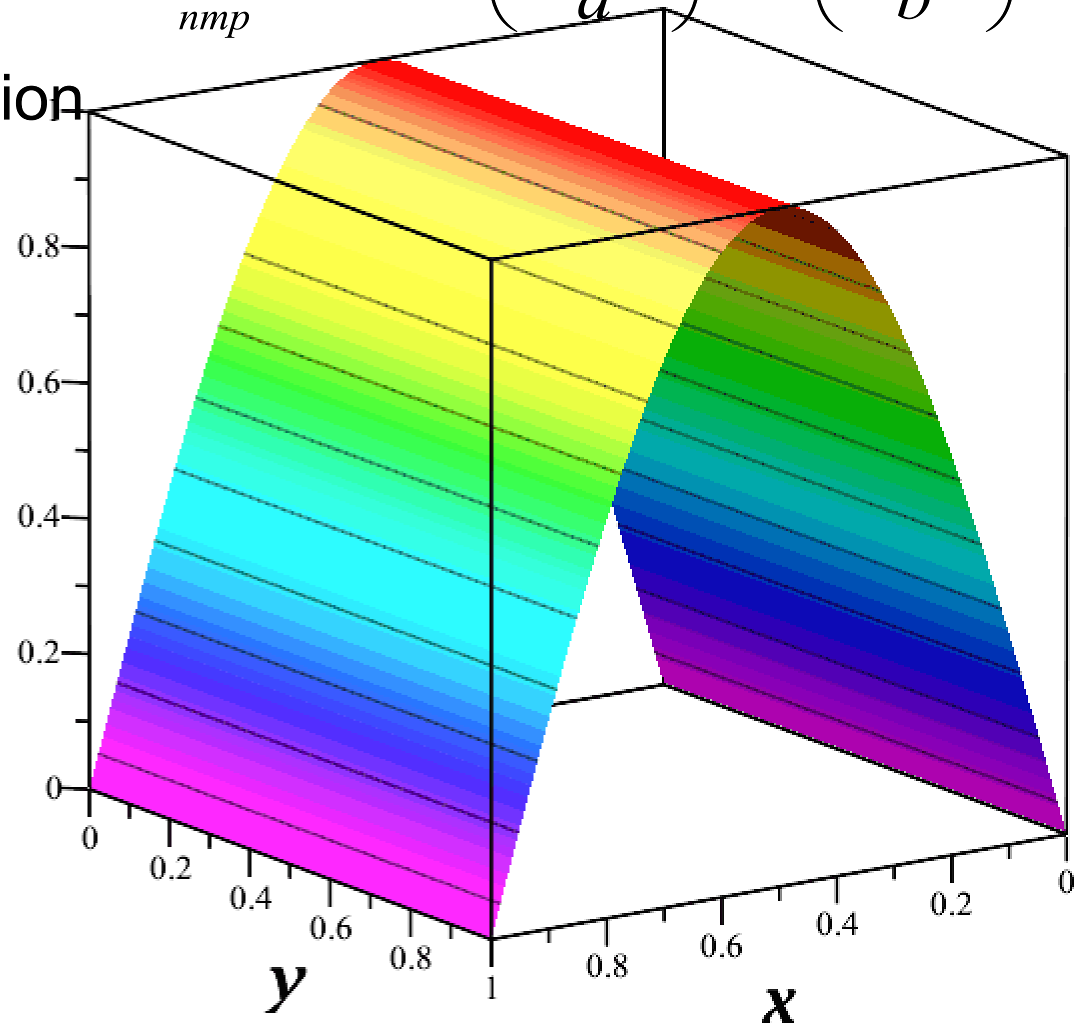
$m=1, n=1, p=0$

Full solution:

$t=0.$

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

Time evolution
nmp=100
at *z=0*



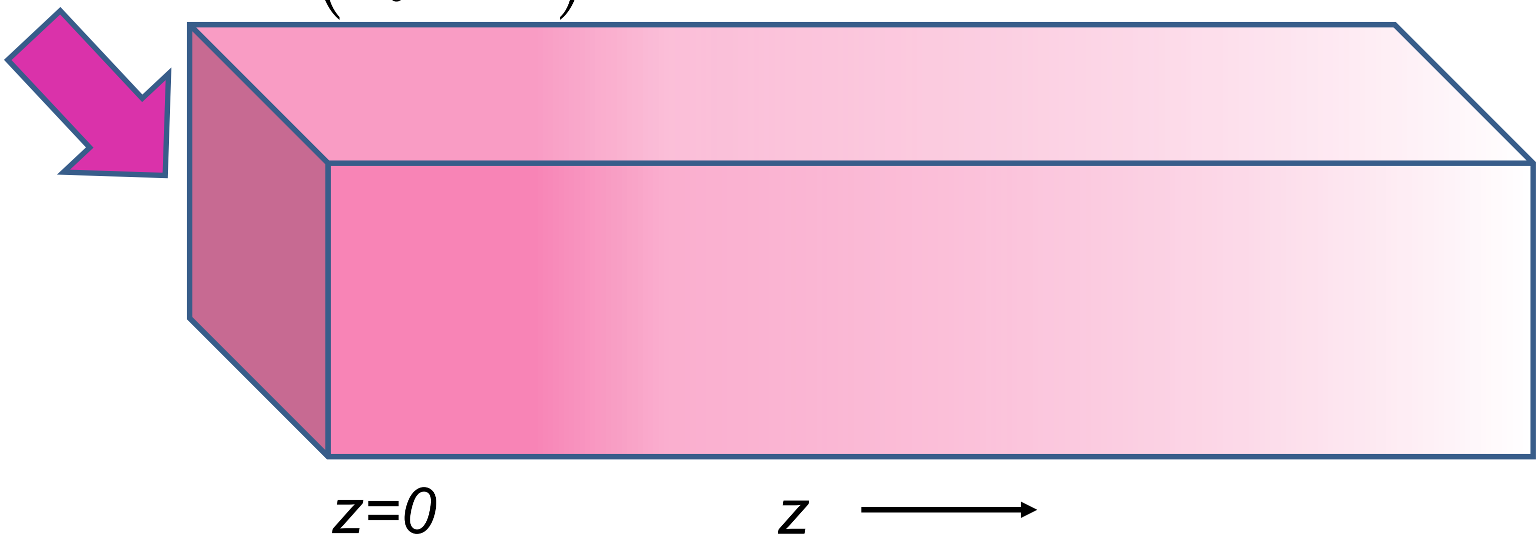
Your question – What real system could have such a temperature distribution

Comment – While one can imagine that the boundary conditions can be readily realized, the single normal mode patterns are much harder. On the other hand, we see that the lowest values of lambda have the longest time constants.

Oscillatory thermal behavior

Here we assume that the spatial variation is along z

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t)=\Re\left(f(z)e^{-i\omega t}\right)$

$$(-i\omega)f=\kappa\frac{d^2 f}{dz^2}$$

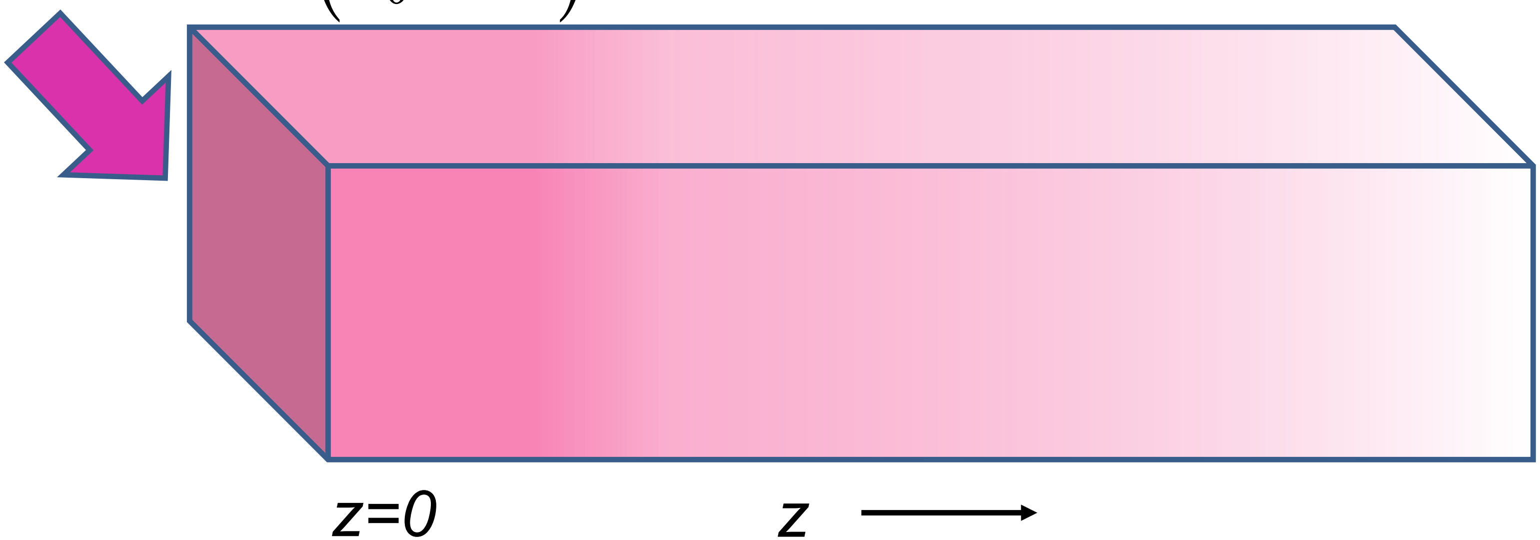
Let $f(z)=Ae^{\alpha z}$

$$\alpha^2=-\frac{i\omega}{\kappa}=e^{3i\pi/2}\;\frac{\omega}{\kappa}$$

$$\alpha=\pm(1-i)\sqrt{\frac{\omega}{2\kappa}}$$

Oscillatory thermal behavior -- continued

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$

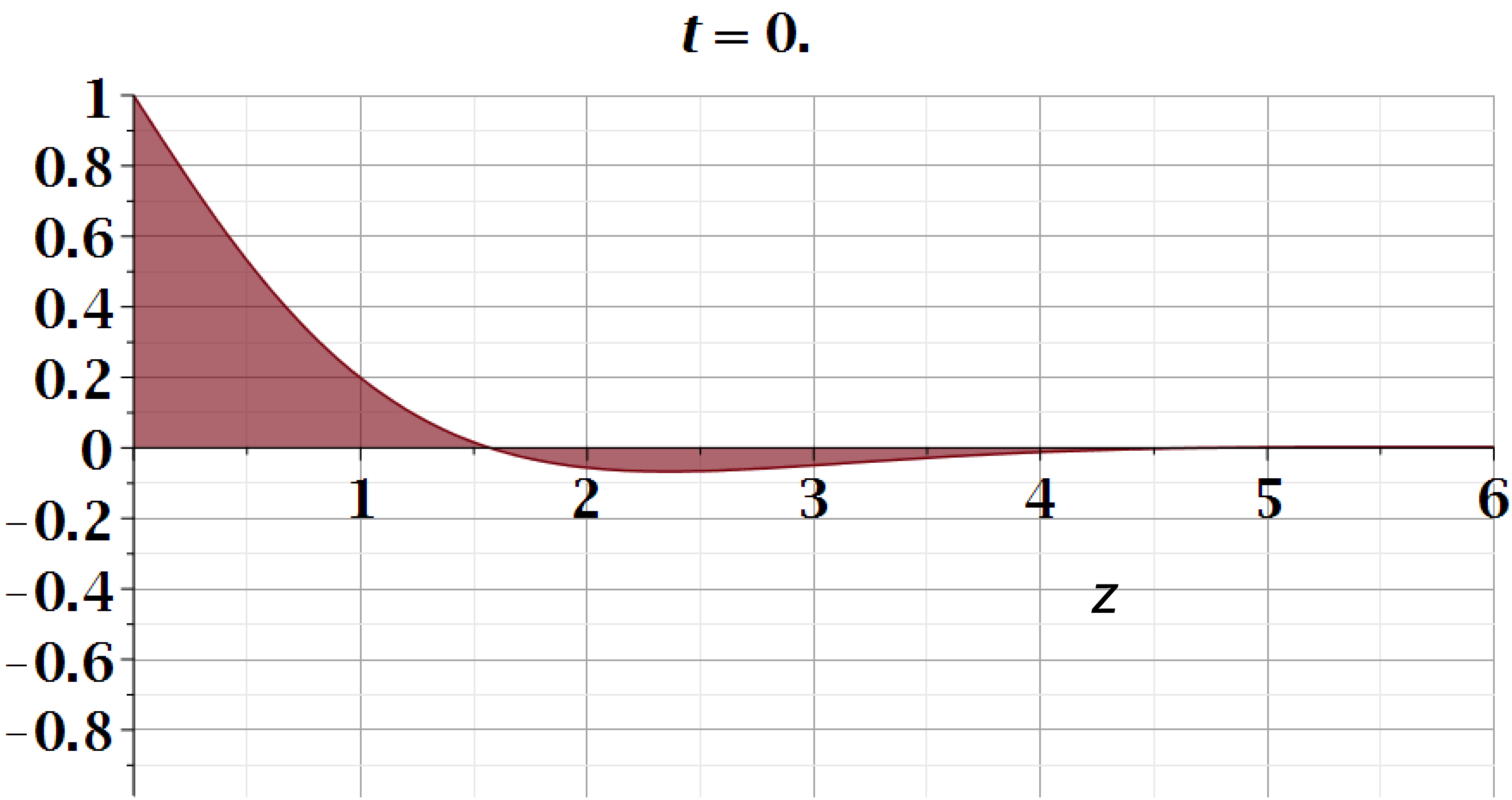


$$T(z,t) = \Re\left(A e^{\pm(1-i)z/\delta} e^{-i\omega t}\right)$$

where $\delta \equiv \sqrt{\dfrac{2\kappa}{\omega}}$

Physical solution: $T(z,t) = T_0 e^{-z/\delta} \cos\left(\dfrac{z}{\delta} - \omega t\right)$

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$



Your question – Does this expression say the temperature transmits along the z axis?

Comment – In this case, our setup approximates trivial variation in the x-y plane so that all variation is along z. The spatial form along z with oscillating boundary condition at z=0 is a result of the form of the heat equation.

Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let:} \quad \tilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\tilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \tilde{T}(\mathbf{q},0) = \tilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \tilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t) = \tilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r' \, G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } \; G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3} \int d^3q \, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) \Rightarrow T(z,t) \quad \text{with initial and boundary values:}$$

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution:} \quad T = T_0 \, \text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

Your question -- Can you explain the erf function. I've never really understood it. I'm not sure I've actually ever had someone explain it. I've just seen it appear in places before. https://dlmf.nist.gov/7

§7.2(i) Error Functions

7.2.1
$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt,$$

7.2.2
$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt = 1 - \operatorname{erf} z,$$

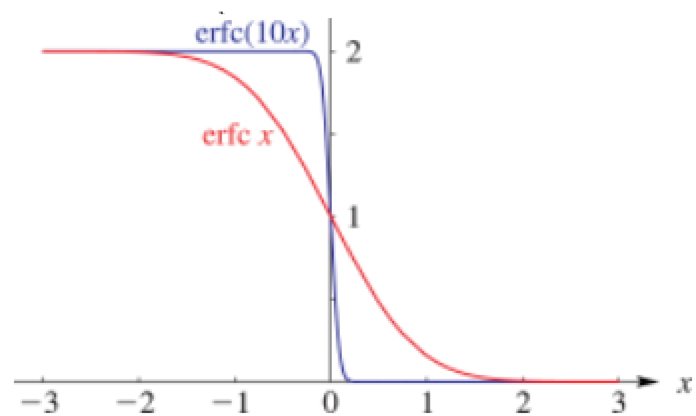


Figure 7.3.1: Complementary error functions $\operatorname{erfc} x$ and $\operatorname{erfc}(10x)$, $-3 \le x \le 3$.

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

$$\text{Solution:} \quad T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

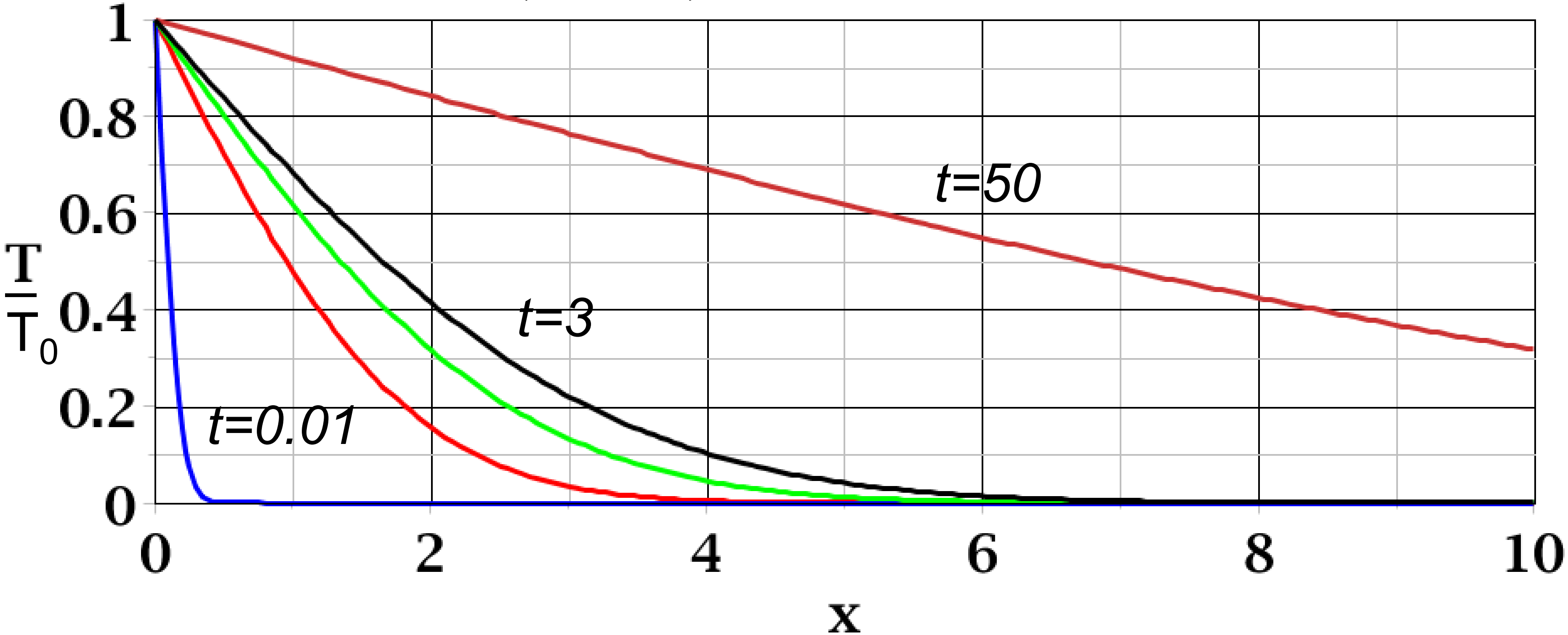
$$\text{where } \operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

$$\text{Note that} \quad \frac{d \operatorname{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$$T = T_0 \ \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$



Temperature profile