PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Plan for Lecture 36: Chap. 11 in F & W

Heat conduction

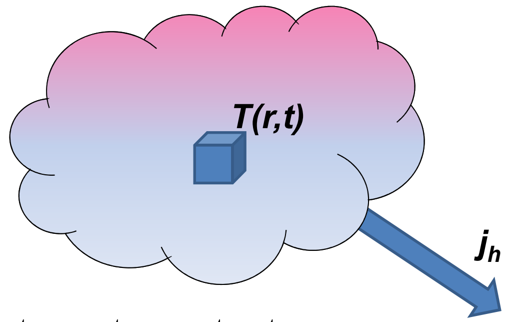
1. Basic equations
2. Boundary value problems

In today's lecture we will take a quick look at heat transfer following Chapter 11 of your textbook.

29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule.

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Enthalpy as a measure of heat of a system at constant pressure in terms of the heat capacity of the material.

Note that in this treatment we are considering a system at constant pressure *p*

Notation: Heat added to system -- $dQ = TdS$

External work done on system -- $dW = -pdV$

Internal energy -- $dE = dQ + dW = TdS - pdV$

Entropy -- dS

Enthalpy -- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \int \rho c_p d^3 r$$

More generally, note that c_p can depend on *T*; we are assuming that dependence to be trivial.

Some notations and concepts from thermodynamics.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Now consider how the enthalpy of a system may change in time. The temperature may change, there may be heat flux, and there may be a source or sink for heat flow.

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

$$\text{Empirically:} \quad \mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

$$\kappa \equiv \frac{k_{th}}{\rho c_p}$$ thermal diffusivity

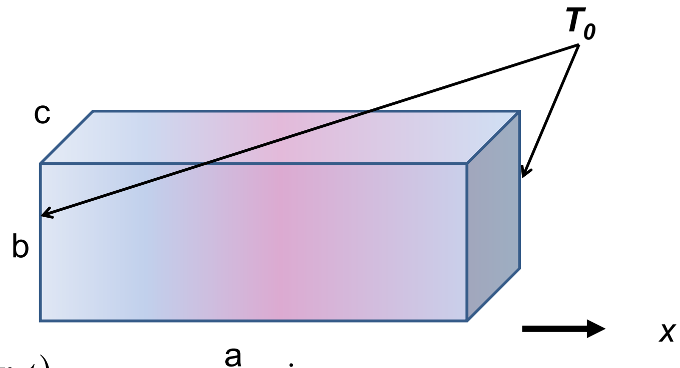
https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm

Typical values (m^2/s)

Air	$2\text{x}10^{-5}$
Water	$1\text{x}10^{-7}$
Copper	$1\text{x}10^{-4}$

In order to relate these quantities, we need to know how enthalpy is related to temperature and we will use the empirical relations based on observation that heat flux is proportional to the gradient of temperature. The Thermal diffusivity coefficient is highly dependent on the material as seen in this short list taken from the internet.

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Example boundary value problem which we will solve in the case that the source term is zero.

Boundary value problems for heat conduction

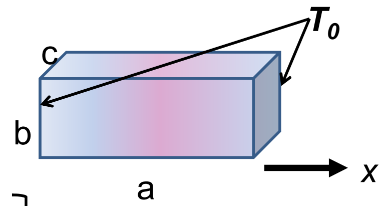
$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

Assuming thermally insulated boundaries



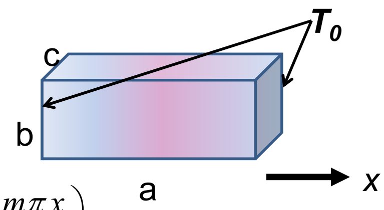
Separation of varables : $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa\left(\alpha^2 + \beta^2 + \gamma^2\right) = 0$$

Using separation of variables to solve the problem.

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

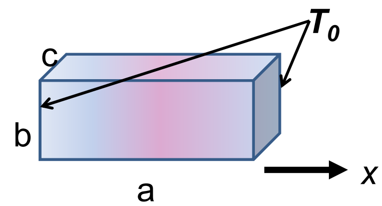
$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Some details for this case.

Boundary value problems for heat conduction



Full solution:

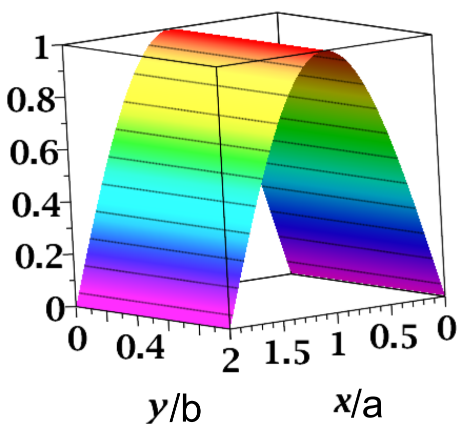
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

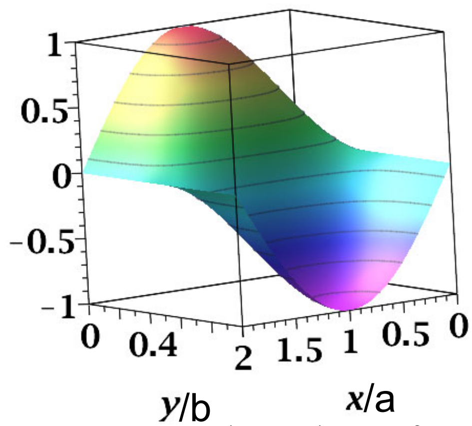
More details.

Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp} t}$$



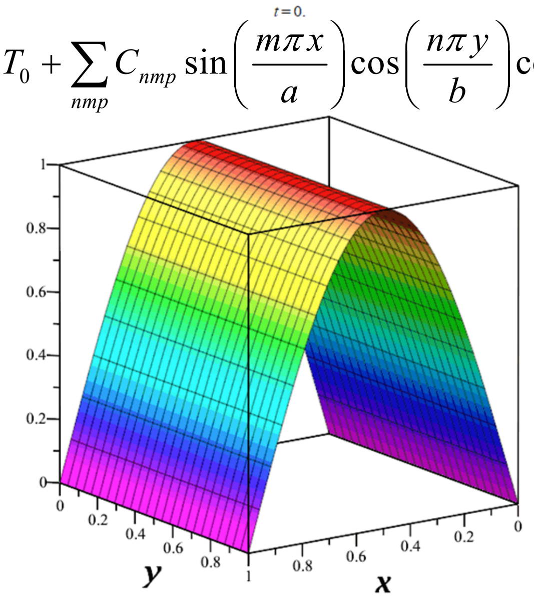
$m=1, n=0, p=0$



$m=1, n=1, p=0$

Full solution:

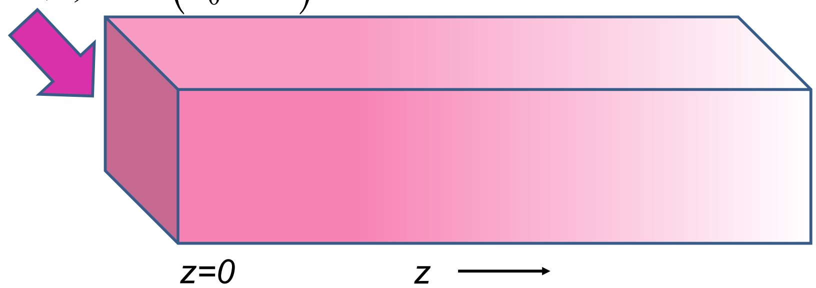
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$



Visualization of the time evolution.

Oscillatory thermal behavior

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t)=\Re\left(f(z)e^{-i\omega t}\right)$

$$\left(-i\omega\right)f=\kappa\frac{d^2 f}{dz^2}$$

Let $f(z)=Ae^{\alpha z}$

$$\alpha^2=-\frac{i\omega}{\kappa}=e^{3i\pi/2}\ \frac{\omega}{\kappa}$$

$$\alpha=\pm\left(1-i\right)\sqrt{\frac{\omega}{2\kappa}}$$

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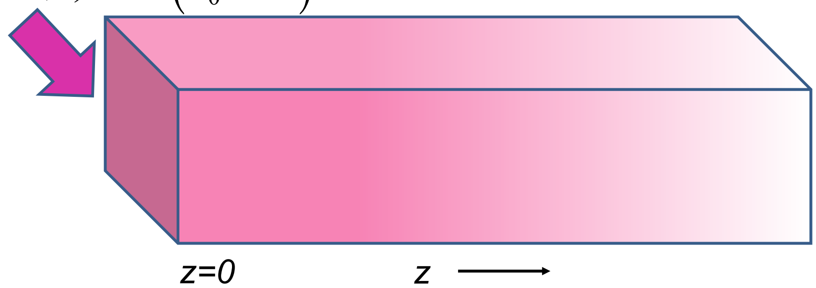
Now consider an oscillatory solutions.

Oscillatory thermal behavior -- continued

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$



$$T(z,t) = \Re\left(A e^{\pm(1-i)z/\delta} e^{-i\omega t}\right)$$

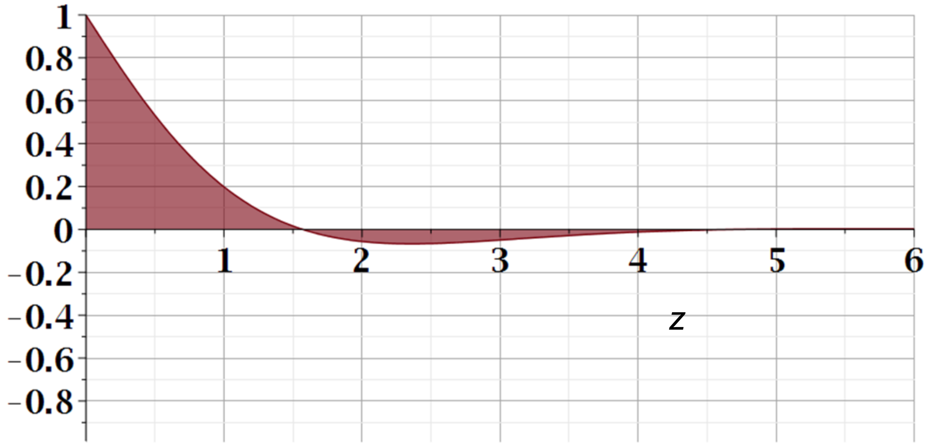
where $\delta \equiv \sqrt{\dfrac{2\kappa}{\omega}}$

Physical solution: $T(z,t) = T_0 e^{-z/\delta} \cos\left(\dfrac{z}{\delta} - \omega t\right)$

Analysis of solution.

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$

$t = 0.$



Animation of solution.

Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let}: \quad \tilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\tilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \tilde{T}(\mathbf{q},0) = \tilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \tilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t) = \tilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Now consider an initial value problem.

Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t)=\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t)=\widetilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\widetilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0)=\widetilde{f}(\mathbf{q})=\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}f(\mathbf{r})$$

$$T(\mathbf{r},t)=\int d^3r' G(\mathbf{r}-\mathbf{r}',t)T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t)\equiv\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}e^{-\kappa q^2 t}$$

Using Green's functions to analyze the results.

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r'\, G(\mathbf{r}-\mathbf{r}',t)\, T(\mathbf{r}',0)$$

$$\text{with}\quad G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}}\, e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Some details.

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$T(\mathbf{r},t) \Rightarrow T(z,t)$ with initial and boundary values :

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution :} \quad T = T_0 \, \text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

For half space boundary.

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

Solution : $\quad T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}} \right)$

where $\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

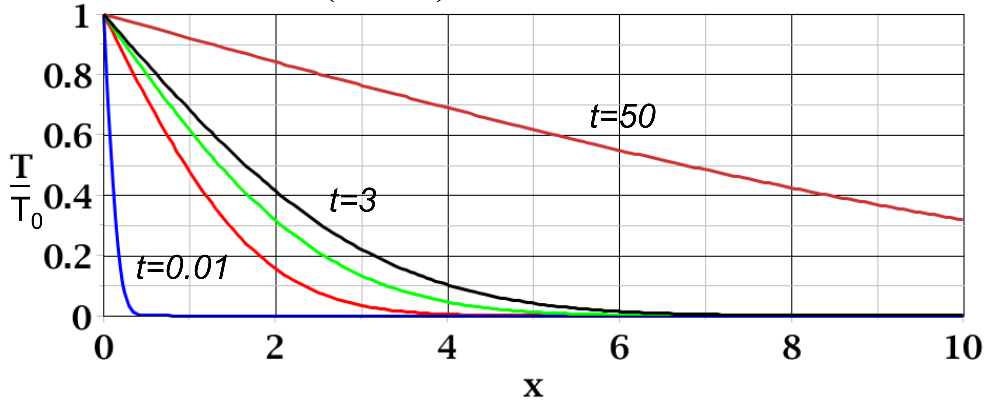
Note that $\quad \frac{d \operatorname{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$

$$\frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}} \right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}} \right)$$

$$\frac{\partial^2}{\partial z^2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}} \right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}} \right)$$

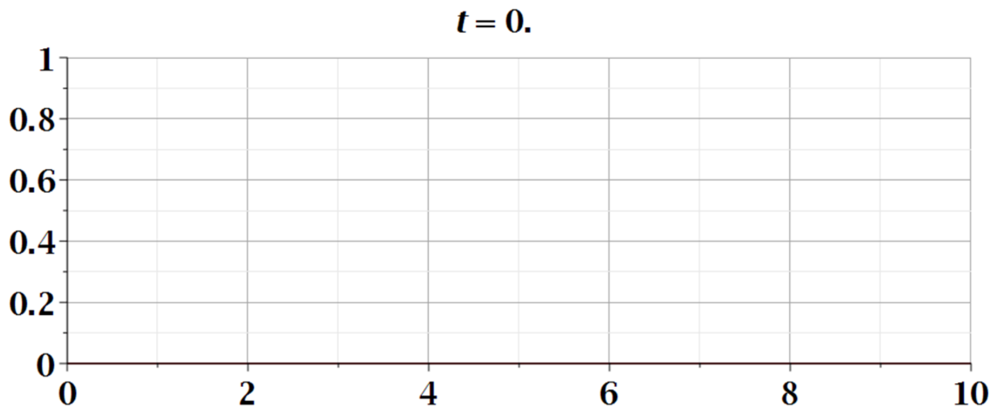
Some details.

$$T = T_0 \ \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$



Plots of solution at various times.

Temperature profile



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Animation.