

PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF online

Discussion for Lecture 37: Chap. 12 in F & W

Viscous fluids

- 1. Viscous stress tensor**
- 2. Navier-Stokes equation**
- 3. Example for incompressible fluid –
Stokes “law”**

Schedule for weekly one-on-one meetings (EST)

Nick – 11 AM Monday

Gao – 9 PM Tuesday

Tim – 11 AM Wednesday

Jeanette – 11 AM Friday

Derek – 12 PM Friday

Your questions –

From Gao –

1, Why can we let $\mathbf{v}$ be like this?

$$\mathbf{v} = \nabla \times (\nabla \times f(r) \mathbf{u}) + \mathbf{u}$$

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
 37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

**Thursday, Nov. 19, 2020
4 PM**



Martha-Elizabeth Baylor, PhD

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Chair of Physics and Astronomy
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Northfield, MN

“A Dynamical System Approach to the Cocktail Party Problem: Using Optics Instead of your Brain to Separate Signals”

Have you ever been at a noisy party and still been able to pick out what the person in front of you is saying? If so, then you are intimately aware of the fact that your brain is able to solve the cocktail party problem. How does your brain separate one signal from a mixture of signals? I have no idea, but I will tell you about a half optical, half electronic system that is able to mimic that behavior. The optoelectronic system uses dynamic holography combined with non-linear electro-optics in a feedback loop to solve the cocktail party problem. By

Equations for motion of non-viscous fluid

Newton-Euler equation of motion:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \mathbf{v} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) = 0$$

Add the two equations:

$$\underbrace{\rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \mathbf{v}}_{\frac{\partial(\rho \mathbf{v})}{\partial t}} + \underbrace{\rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v})}_{\sum_{j=1}^3 \frac{\partial(\rho v_j \mathbf{v})}{\partial x_j}} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

Equations for motion of non-viscous fluid -- continued

Newton-Euler equation in terms of fluid momentum:

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_j \mathbf{v})}{\partial x_j} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_j \mathbf{v})}{\partial x_j} + \nabla p = \rho \mathbf{f}_{\text{applied}}$$

Fluid momentum: $\rho \mathbf{v}$

Stress tensor: $T_{ij} \equiv \rho v_i v_j + p \delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{j=1}^3 \frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

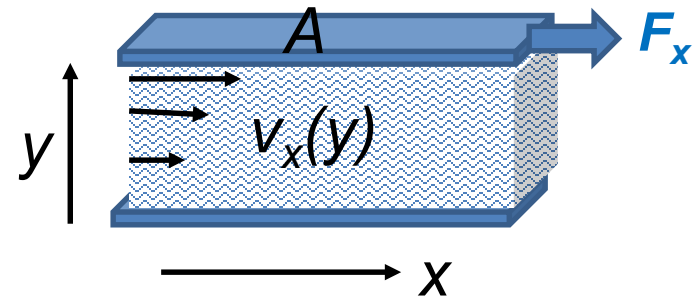
$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p \delta_{ij} = T_{ji}^{\text{ideal}}$$

As an example of a viscous effect, consider --

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$

material dependent parameter



Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = \underbrace{-\eta \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3} \delta_{kl} (\nabla \cdot \mathbf{v}) \right)}_{\text{viscosity}} - \underbrace{\zeta \delta_{kl} (\nabla \cdot \mathbf{v})}_{\text{bulk viscosity}}$$

Total stress tensor: $T_{kl} = T_{kl}^{\text{ideal}} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p \delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3} \delta_{kl} (\nabla \cdot \mathbf{v}) \right) - \zeta \delta_{kl} (\nabla \cdot \mathbf{v})$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{j=1}^3 \frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_i v_j)}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \eta \sum_{j=1}^3 \frac{\partial^2 v_i}{\partial x_j^2} + \left(\zeta + \frac{1}{3} \eta \right) \sum_{j=1}^3 \frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^3 \frac{\partial(\rho v_j)}{\partial x_j} = 0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v} + \frac{1}{\rho} \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{v})$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v} + \frac{1}{\rho} \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Typical viscosities at 20° C and 1 atm:

Fluid	η/ρ (m ² /s)	η (Pa s)
Water	1.00 x 10 ⁻⁶	1 x 10 ⁻³
Air	14.9 x 10 ⁻⁶	0.018 x 10 ⁻³
Ethyl alcohol	1.52 x 10 ⁻⁶	1.2 x 10 ⁻³
Glycerine	1183 x 10 ⁻⁶	1490 x 10 ⁻³

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v} + \frac{1}{\rho} \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\text{Note that } \nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

$$\text{Incompressible fluid} \Rightarrow \nabla \cdot \mathbf{v} = 0$$

$$\text{Steady flow} \Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$$

$$\text{Irrotational flow} \Rightarrow \nabla \times \mathbf{v} = 0$$

$$\text{No applied force} \Rightarrow \mathbf{f} = 0$$

$$\text{Neglect non-linear terms} \Rightarrow \nabla (v^2) = 0$$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

Navier-Stokes equation becomes:

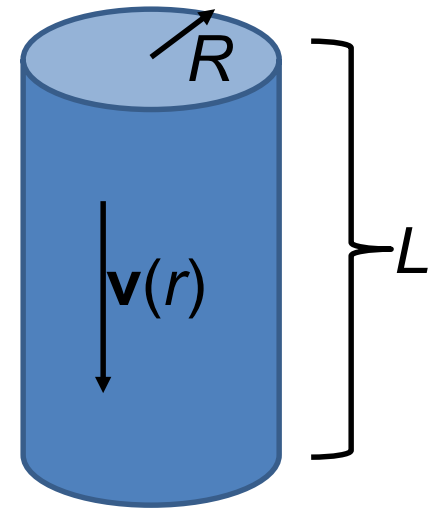
$$0 = -\frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v}$$

Assume that $\mathbf{v}(\mathbf{r}, t) = v_z(r) \hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta \nabla^2 v_z(r) \quad (\text{independent of } z)$$

Suppose that $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$ (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$



Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

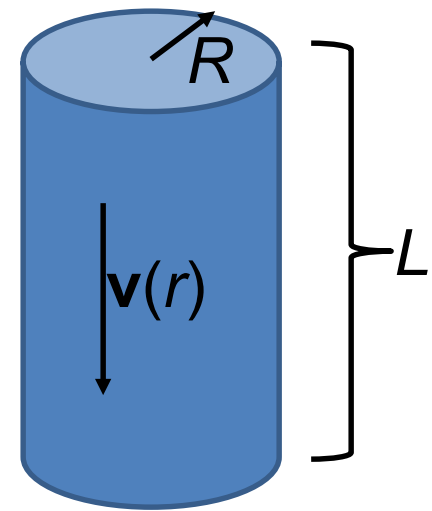
$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r} \frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$$\Rightarrow C_1 = 0 \quad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_2$$

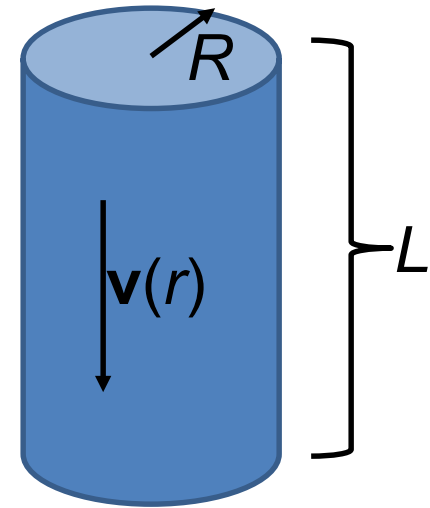
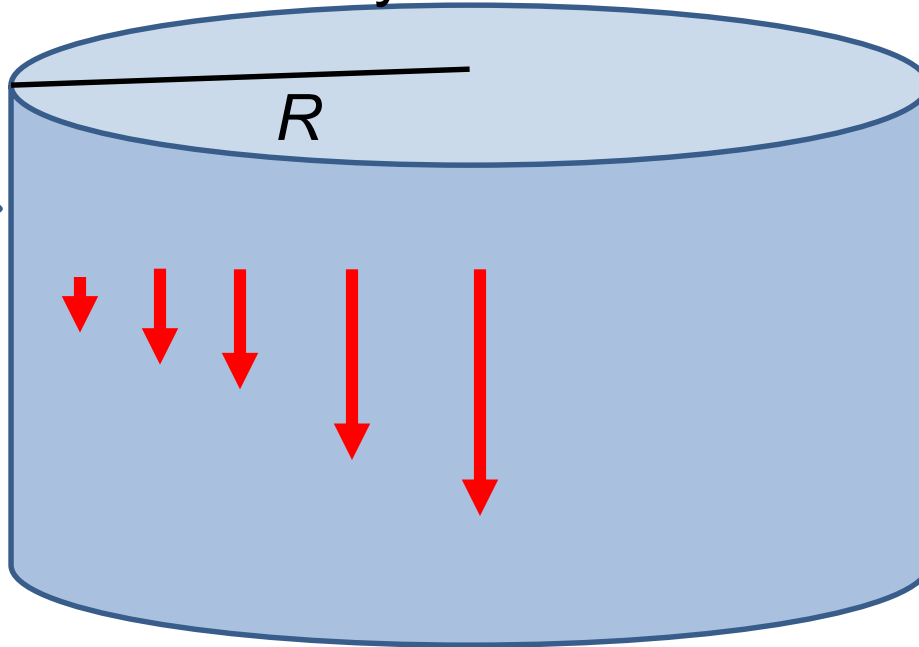
$$v_z(r) = \frac{\Delta p}{4\eta L} (R^2 - r^2)$$



Comment on boundary condition

$$v_z(R) = 0$$

Fluid approximately stationary
at boundary



Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

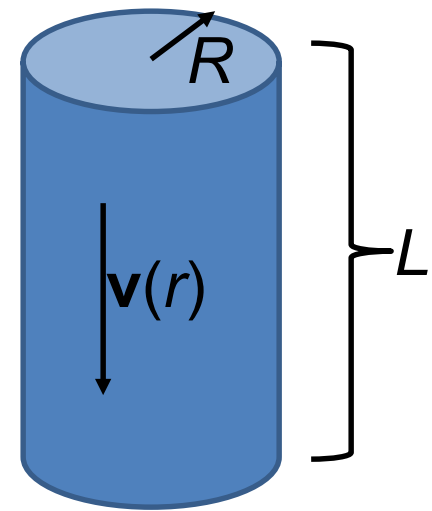
$$v_z(r) = \frac{\Delta p}{4\eta L} (R^2 - r^2)$$

Mass flow rate through the pipe:

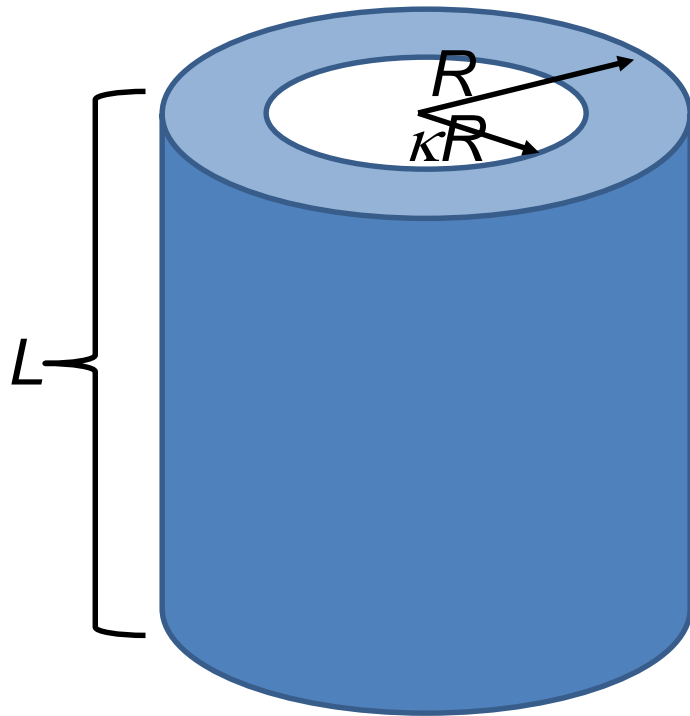
$$\frac{dM}{dt} = 2\pi\rho \int_0^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$

Poiseuille formula;

➔ Method for measuring η



Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius R and inner radius κR



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r} \frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

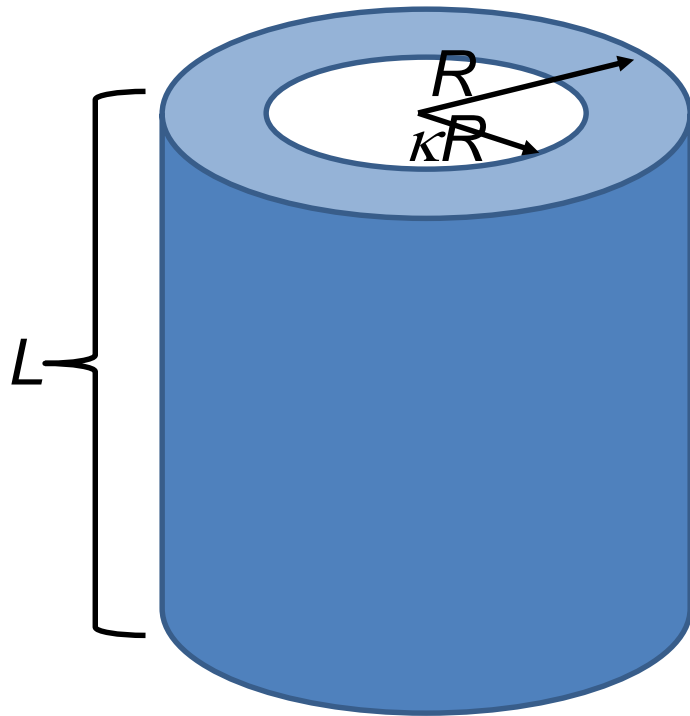
$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + C_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + C_2$$

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius R and inner radius κR -- continued

Solving for C_1 and C_2 :

$$v_z(r) = \frac{\Delta p R^2}{4\eta L} \left(1 - \left(\frac{r}{R} \right)^2 - \frac{1 - \kappa^2}{\ln \kappa} \ln \left(\frac{r}{R} \right) \right)$$



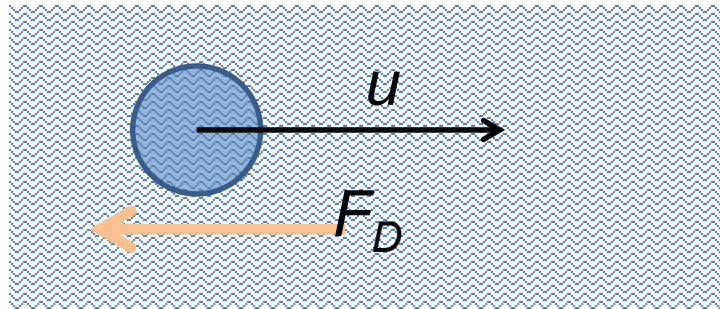
Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho \int_{\kappa R}^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L} \left(1 - \kappa^4 + \frac{(1 - \kappa^2)^2}{\ln \kappa} \right)$$

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η :

$$F_D = -\eta(6\pi Ru)$$



Plan:

1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

Have you ever encountered Stokes law in previous contexts?

- a. Milliken oil drop experiment
- b. A sphere falling due to gravity in a viscous fluid, reaching a terminal velocity
- c. Other?

Newton-Euler equation for incompressible fluid,
modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} + \underbrace{\frac{\eta}{\rho}}_{\nu} \nabla^2 \mathbf{v}$$

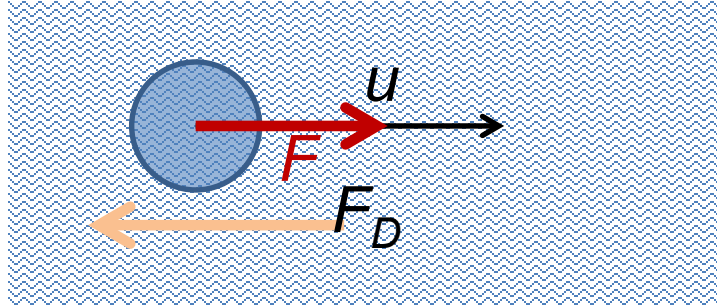
Kinematic viscosity

Typical kinematic viscosities at 20° C and 1 atm:

Fluid	ν (m ² /s)
Water	1.00 x 10 ⁻⁶
Air	14.9 x 10 ⁻⁶
Ethyl alcohol	1.52 x 10 ⁻⁶
Glycerine	1183 x 10 ⁻⁶

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η :

$$F_D = -\eta(6\pi R u)$$



Effects of drag force on motion of
particle of mass m with constant force F :

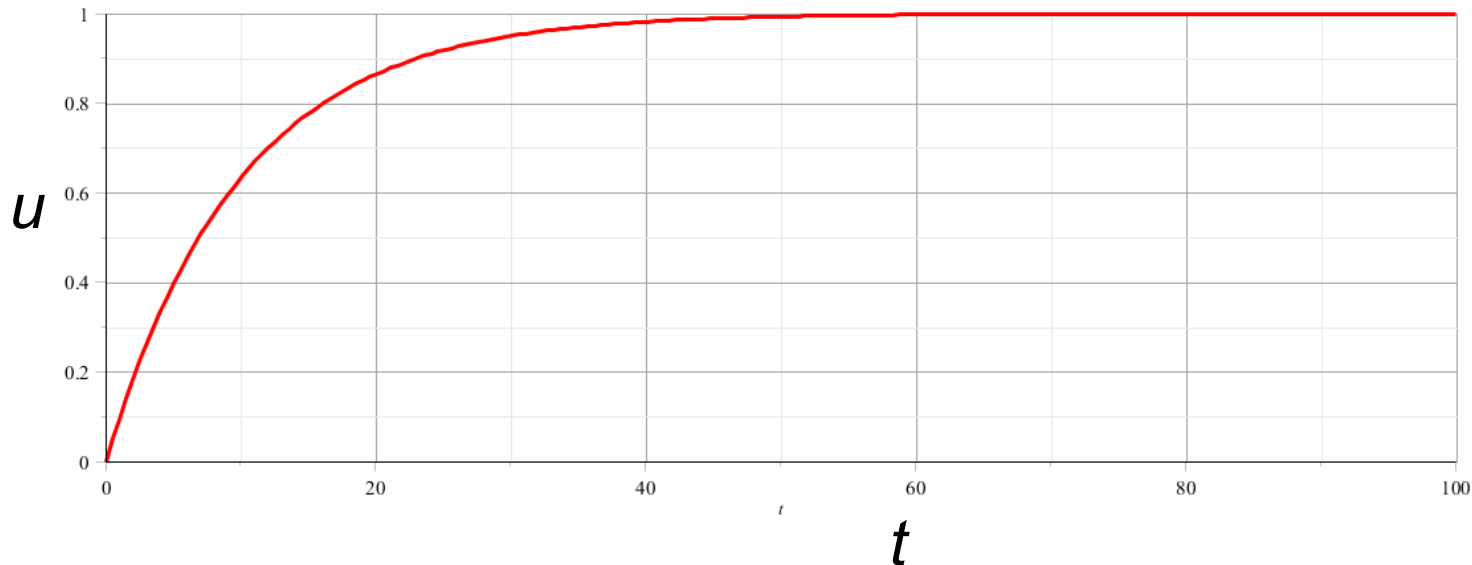
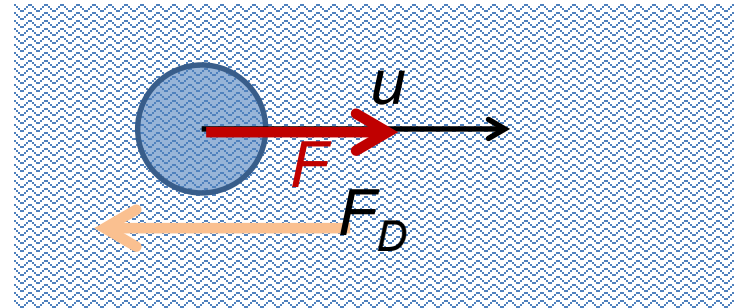
$$F - 6\pi R \eta u = m \frac{du}{dt} \quad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R \eta} \left(1 - e^{-\frac{6\pi R \eta}{m} t} \right)$$

Effects of drag force on motion of
particle of mass m with constant force F :

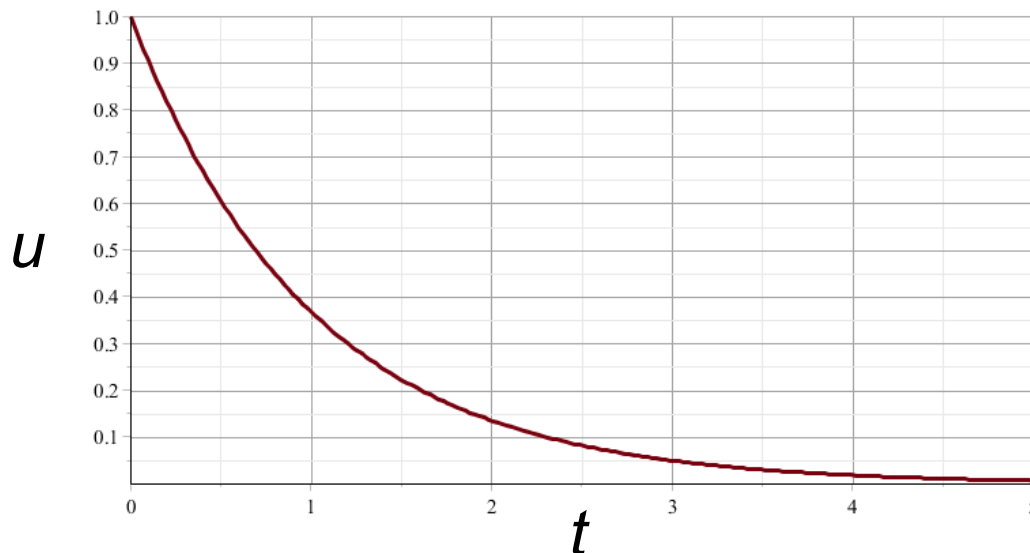
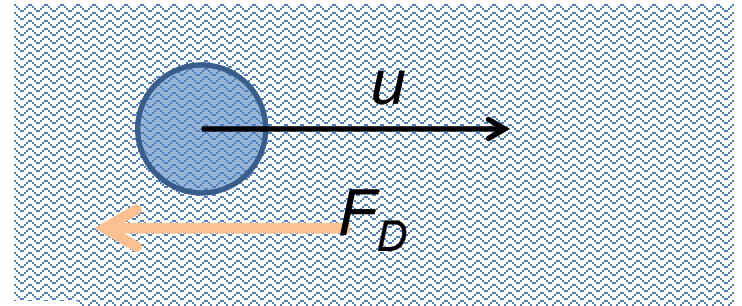
$$F - 6\pi R \eta u = m \frac{du}{dt} \quad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R \eta} \left(1 - e^{-\frac{6\pi R \eta}{m} t} \right)$$



Effects of drag force on motion of particle of mass m
with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m \frac{du}{dt}$$
$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$



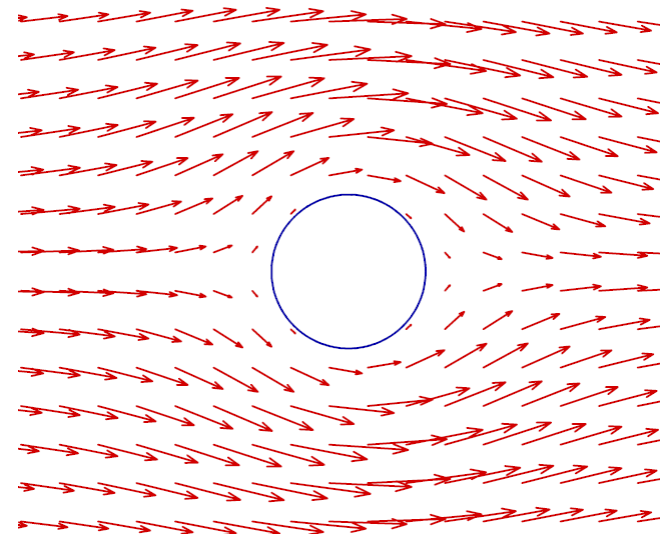
Recall: PHY 711 -- Assignment #18 Oct. 26, 2020

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the $\mathbf{z}$ direction at large distances from a spherical obstruction of radius a . Find the form of the velocity potential and the velocity field for all $r > a$. Assume that for $r = a$, the velocity in the radial direction is 0 but the velocity in the azimuthal direction is not necessarily 0.

$$\nabla^2 \Phi = 0$$

$$\Phi(r, \theta) = -v_0 \left(r + \frac{a^3}{2r^2} \right) \cos \theta$$

In the present viscous case, we will assume that $\mathbf{v}(a)=0$.



Newton-Euler equation for incompressible fluid,
modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{v}$$

Continuity equation: $\nabla \cdot \mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{\text{applied}} = 0$;

$$\Rightarrow \nabla p = \eta \nabla^2 \mathbf{v}$$

$$\nabla p = \eta \nabla^2 \mathbf{v}$$

Take curl of both sides of equation:

$$\nabla \times (\nabla p) = 0 = \eta \nabla^2 (\nabla \times \mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times (\nabla \times f(r) \mathbf{u}) + \mathbf{u}$$

$$\text{where } f(r) \xrightarrow{r \rightarrow \infty} 0$$

Note that:

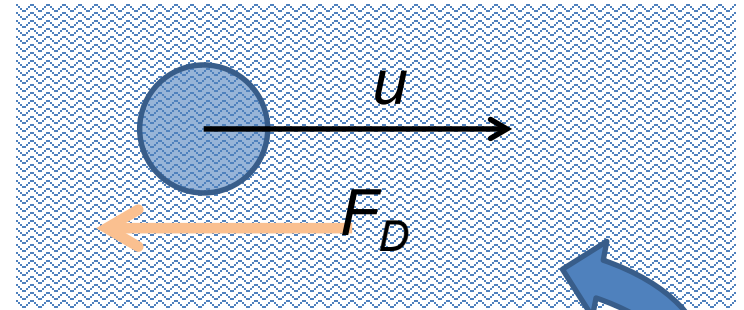
$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Your question – why assume

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times (\nabla \times f(r) \mathbf{u}) + \mathbf{u}$$

where $f(r) \xrightarrow{r \rightarrow \infty} 0$



$\mathbf{v}(r, \theta) \approx \mathbf{u}$

Digression

Some comment on assumption: $\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Here $\mathbf{A} = f(r)\mathbf{u}$

$$\nabla \times \mathbf{v} = \nabla \times (\nabla \times (\nabla \times \mathbf{A})) = -\nabla \times (\nabla^2 \mathbf{A})$$

Also note: $\nabla p = \eta \nabla^2 \mathbf{v}$

$$\Rightarrow \nabla \times \nabla p = \nabla \times \eta \nabla^2 \mathbf{v} \quad \text{or} \quad \nabla^2 (\nabla \times \mathbf{v}) = 0$$

$$\nabla^2 (\nabla \times \nabla^2 \mathbf{A}) = \nabla^4 (\nabla \times \mathbf{A}) = 0$$

$$\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla \times (\nabla \times f(r)\hat{\mathbf{z}}) = \nabla(\nabla \cdot f(r)\hat{\mathbf{z}}) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = 0 \quad \Rightarrow \quad \nabla^2 (\nabla \times \mathbf{v}) = 0$$

$$\nabla^4 (\nabla \times f(r)\hat{\mathbf{z}}) = 0 \quad \Rightarrow \quad \nabla^4 (\nabla f(r) \times \hat{\mathbf{z}}) = 0 \quad \Rightarrow \quad \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u \cos \theta \left(1 - \frac{2}{r} \frac{df}{dr} \right) = u \cos \theta \left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3} \right)$$

$$v_\theta = -u \sin \theta \left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right) = -u \sin \theta \left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3} \right)$$

Some details:

$$\nabla^4 f(r) = 0 \quad \Rightarrow \quad \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right)^2 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$\begin{aligned} \mathbf{v} &= u \left(\nabla \times \left(\nabla \times f(r) \hat{\mathbf{z}} \right) + \hat{\mathbf{z}} \right) \\ &= u \left(\nabla \left(\nabla \cdot \left(f(r) \hat{\mathbf{z}} \right) \right) - \nabla^2 f(r) \hat{\mathbf{z}} + \hat{\mathbf{z}} \right) \end{aligned}$$

Note that: $\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$

$$\mathbf{v} = u \left(\nabla \left(\frac{df}{dr} \cos \theta \right) - \left(\nabla^2 (f(r)) - 1 \right) \left(\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \right) \right)$$

$$v_r = u \cos \theta \left(1 - \frac{2}{r} \frac{df}{dr} \right) = u \cos \theta \left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3} \right)$$

$$v_\theta = -u \sin \theta \left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right) = -u \sin \theta \left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3} \right)$$

To satisfy $\mathbf{v}(r \rightarrow \infty) = \mathbf{u}$: $\Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R) = 0$ solve for C_2, C_4

$$v_r = u \cos \theta \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right)$$

$$v_\theta = -u \sin \theta \left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3} \right)$$

$$v_r = u \cos \theta \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right)$$

$$v_\theta = -u \sin \theta \left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3} \right)$$

Determining pressure:

$$\nabla p = \eta \nabla^2 \mathbf{v} = -\eta \nabla \left(u \cos \theta \left(\frac{3R}{2r^2} \right) \right)$$

$$\Rightarrow p(r) = p_0 - \eta u \cos \theta \left(\frac{3R}{2r^2} \right)$$

$$p(r) = p_0 - \eta u \cos \theta \left(\frac{3R}{2r^2} \right)$$

Corresponds to:

$$F_D \cos \theta = (p(R) - p_0) 4\pi R^2$$

$$\Rightarrow F_D = -\eta u (6\pi R)$$

