PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Plan for Lecture 37: Chap. 12 in F & W

Viscous fluids

1. Viscous stress tensor
2. Navier-Stokes equation
3. Example for incompressible fluid – Stokes "law"

In this lecture, we will consider some effects of viscosity on the motion of fluids, following Chapter 12 of your textbook.

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule.

Wake Forest University

Department of Physics

Colloquium

Thursday, Nov. 19, 2020
4 PM

Martha-Elizabeth Baylor, PhD
Associate Professor of Physics
Chair of Physics and Astronomy
Carleton College
Northfield, MN

"A Dynamical System Approach to the Cocktail Party Problem: Using Optics Instead of your Brain to Separate Signals"

Have you ever been at a noisy party and still been able to pick out what the person in front of you is saying? If so, then you are intimately aware of the fact that your brain is able to solve the cocktail party problem. How does your brain separate one signal from a mixture of signals? I have no idea, but I will tell you about a half optical, half electronic system that is able to mimic that behavior. The optoelectronic system uses dynamic holography combined with non-linear electro-optics in a feedback loop to solve the cocktail party problem. By

The week's colloquium speaker has an intriguing topic --

Equations for motion of non-viscous fluid

Newton-Euler equation of motion:

$$\rho\frac{\partial\mathbf{v}}{\partial t}+\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}=\rho\mathbf{f}_{applied}-\nabla p$$

Continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{v}\right)=0 \qquad \Rightarrow \mathbf{v}\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{v}\right)\right)=0$$

Add the two equations:

$$\underbrace{\rho\frac{\partial\mathbf{v}}{\partial t}+\frac{\partial\rho}{\partial t}\mathbf{v}}_{\frac{\partial\left(\rho\mathbf{v}\right)}{\partial t}}+\underbrace{\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}+\mathbf{v}\nabla\cdot\left(\rho\mathbf{v}\right)}_{\sum_{j=1}^{3}\frac{\partial\left(\rho v_j\mathbf{v}\right)}{\partial x_j}}=\rho\mathbf{f}_{applied}-\nabla p$$

Reviewing the fluid equations we have discussed previously, combining Newton's equations with the continuity equation to find a new convenient form.

Equations for motion of non-viscous fluid -- continued

Newton-Euler equation in terms of fluid momentum:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}+\nabla p=\rho\mathbf{f}_{applied}$$

Fluid momentum: $\rho\mathbf{v}$

Stress tensor: $T_{ij}\equiv\rho v_i v_j + p\delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

Here we recognize terms that have the units of force/area and can be described as a stress tensor Tij.

Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

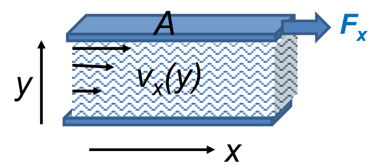
$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p\delta_{ij} = T_{ji}^{\text{ideal}}$$

As an example of a viscous effect, consider --

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$

material dependent parameter



The next step is to imagine that the additional effects of viscosity should/can be represented as a viscous stress tensor. The example of sheer force suggests that the viscous stress tensor involves derivatives of the velocity of the fluid.

Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla \cdot \mathbf{v}\right) \right) - \zeta \delta_{kl}\left(\nabla \cdot \mathbf{v}\right)$$

viscosity

bulk viscosity

Total stress tensor: $T_{kl} = T_{kl}^{\text{ideal}} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla \cdot \mathbf{v}\right) \right) - \zeta \delta_{kl}\left(\nabla \cdot \mathbf{v}\right)$$

Imagining the most general form of the viscous tensor, we consider all derivatives of all components of fluid velocity, separating out the terms with zero trace, with the remaining terms proportional to the divergence of the velocity and representing the “bulk” viscosity.

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial\left(\rho v_i v_j\right)}{\partial x_j}=\rho f_i-\frac{\partial p}{\partial x_i}+\eta\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2}+\left(\zeta+\frac{1}{3}\eta\right)\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity equation

$$\frac{\partial\rho}{\partial t}+\sum_{j=1}^{3}\frac{\partial\left(\rho v_j\right)}{\partial x_j}=0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

Now we can write the fluid equations with the full stress tensor. The continuity equation still applies. The so-called Navier-Stokes equation summarizes the expected behavior of fluids in terms of the material dependent viscosity parameters eta and zera.

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Typical viscosities at 20^o C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

Here is a list of some typical values of the viscosity parameter eta.

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R*

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Note that $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

Incompressible fluid $\Rightarrow$ $\nabla \cdot \mathbf{v} = 0$

Steady flow $\Rightarrow$ $\frac{\partial \mathbf{v}}{\partial t} = 0$

Irrotational flow $\Rightarrow$ $\nabla \times \mathbf{v} = 0$

No applied force $\Rightarrow$ $\mathbf{f} = 0$

Neglect non-linear terms $\Rightarrow \nabla(v^2) = 0$

Example of a measurement of viscosity for irrotational flow.

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

Navier-Stokes equation becomes:

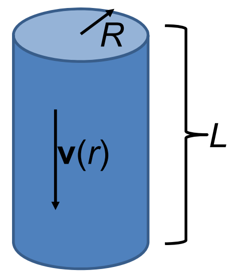
$$0 = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v}$$

Assume that $\mathbf{v}(\mathbf{r},t) = v_z(r)\hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta \nabla^2 v_z(r) \quad \text{(independent of } z\text{)}$$

Suppose that $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$ (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$



Continued analysis of simple viscous flowl

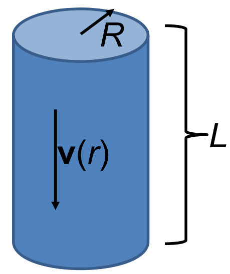
Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + \mathrm{C}_2$$

$$\Rightarrow C_1 = 0 \qquad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + \mathrm{C}_2$$



$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

11/18/2020 PHY 711 Fall 2020 -- Lecture 37 12

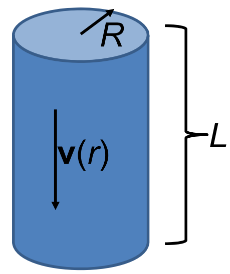
Solving for the velocity profile.

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho\int_0^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$

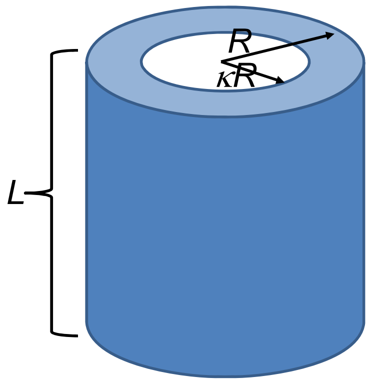


Poiseuille formula;

➔Method for measuring η

This analysis is useful for measuring eta.

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius *κR*



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

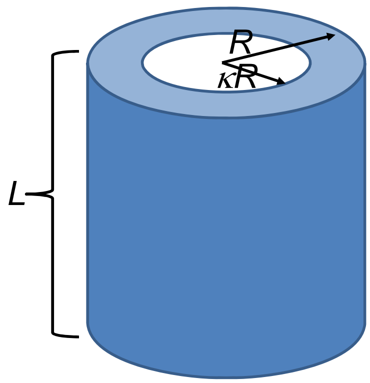
$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + \mathrm{C}_2$$

$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + \mathrm{C}_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + \mathrm{C}_2$$

Another related system with a cylindrical shell.

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius κR -- continued



Solving for C_1 and C_2 :

$$v_z(r) = \frac{\Delta p R^2}{4\eta L}\left(1 - \left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{r}{R}\right)\right)$$

Mass flow rate through the pipe:

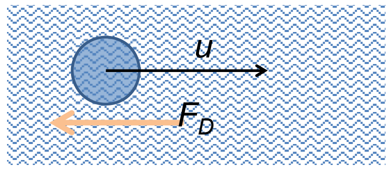
$$\frac{dM}{dt} = 2\pi\rho\int_{\kappa R}^{R} r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}\left(1 - \kappa^4 + \frac{\left(1-\kappa^2\right)^2}{\ln\kappa}\right)$$

The final result again can be used to measure the viscosity.

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta(6\pi R u)$$



Plan:
1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

Changing to an analysis of viscous flow as a drag force.

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \underbrace{\frac{\eta}{\rho}}_{\upsilon}\nabla^2\mathbf{v}$$

Kinematic viscosity

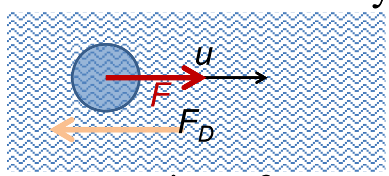
Typical kinematic viscosities at 20° C and 1 atm:

Fluid	ν (m^2/s)
Water	1.00 x 10^{-6}
Air	14.9 x 10^{-6}
Ethyl alcohol	1.52 x 10^{-6}
Glycerine	1183 x 10^{-6}

In this case, we will consider an incompressible fluid in which case eta/rho is the important parameter.

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η :

$$F_D = -\eta(6\pi R u)$$



Effects of drag force on motion of particle of mass m with constant force F :

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

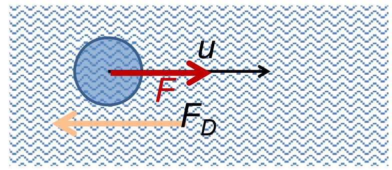
$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$

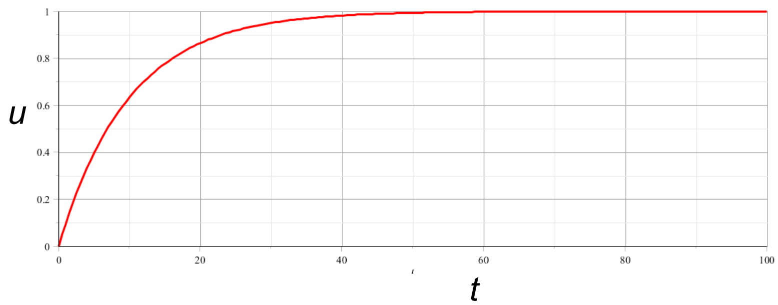
Before deriving Stokes law of viscous drag, it is interesting to recall its effects.

Effects of drag force on motion of
particle of mass m with constant force F :

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$





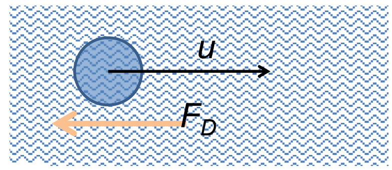
Objects moving in the presence of the Stokes viscous drag, tend to read a steady “terminal” velocity.

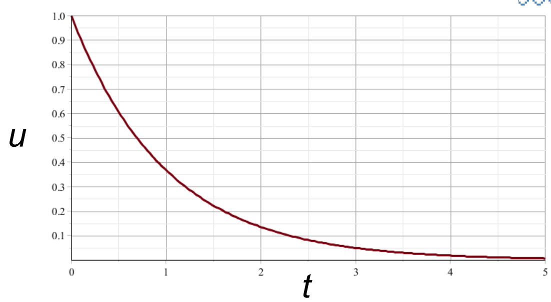
Effects of drag force on motion of particle of mass m

with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m\frac{du}{dt}$$

$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$





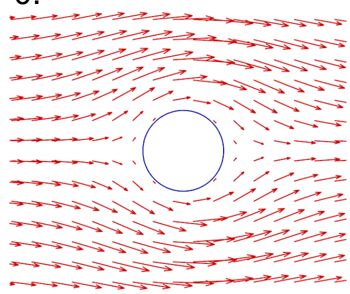
Or the velocity decays to zero.

Recall: PHY 711 -- Assignment #18 Oct. 26, 2020

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the **z** direction at large distances from a spherical obstruction of radius *a*. Find the form of the velocity potential and the velocity field for all *r* > *a*. Assume that for *r* = *a*, the velocity in the radial direction is 0 but the velocity in the azimuthal direction is not necessarily 0.

$$\nabla^2\Phi = 0$$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$



In previous discussions without viscosity, the velocity near the sphere is not necessarily zero. How will this be affected in the presence of viscosity?

Newton-Euler equation for incompressible fluid,
modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2 \mathbf{v}$$

Continuity equation: $\nabla \cdot \mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{applied} = 0$;

$$\Rightarrow \nabla p = \eta \nabla^2 \mathbf{v}$$

Here we keep the dominant terms, finding a relationship between the pressure and the velocity.

$$\nabla p = \eta \nabla^2 \mathbf{v}$$

Take curl of both sides of equation:

$$\nabla \times (\nabla p) = 0 = \eta \nabla^2 (\nabla \times \mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$$

where $f(r) \xrightarrow[r\to\infty]{} 0$

Note that:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

This analysis follows the treatment of Landau and Lifshitz.

Digression

Some comment on assumption: $\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u}\right) + \mathbf{u}$

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A}$$

Here $\mathbf{A} = f(r)\mathbf{u}$

$$\nabla \times \mathbf{v} = \nabla \times \left(\nabla \times \left(\nabla \times \mathbf{A}\right)\right) = -\nabla \times \left(\nabla^2 \mathbf{A}\right)$$

Also note: $\nabla p = \eta \nabla^2 \mathbf{v}$

$$\Rightarrow \nabla \times \nabla p = \nabla \times \eta \nabla^2 \mathbf{v} \qquad \text{or} \qquad \nabla^2\left(\nabla \times \mathbf{v}\right) = 0$$

Deducing the form of the velocity

$$\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u}\right) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla \times \left(\nabla \times f(r)\hat{\mathbf{z}}\right) = \nabla\left(\nabla \cdot f(r)\hat{\mathbf{z}}\right) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \nabla^2\left(\nabla \times \mathbf{v}\right) = 0$$

$$\nabla^4\left(\nabla \times f(r)\hat{\mathbf{z}}\right) = 0 \quad \Rightarrow \nabla^4\left(\nabla f(r) \times \hat{\mathbf{z}}\right) = 0 \quad \Rightarrow \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u\cos\theta\left(1 - \frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3}\right)$$

Here we find the most general form of the equation that satisfies the differential equation.

Some details:

$$\nabla^4 f(r) = 0 \qquad \Rightarrow \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right)^2 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$\mathbf{v} = u\Big(\nabla\times\big(\nabla\times f(r)\hat{\mathbf{z}}\big) + \hat{\mathbf{z}}\Big)$$

$$= u\Big(\nabla\big(\nabla\cdot\big(f(r)\hat{\mathbf{z}}\big)\big) - \nabla^2 f(r)\hat{\mathbf{z}} + \hat{\mathbf{z}}\Big)$$

Note that: $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

$$\mathbf{v} = u\left(\nabla\left(\frac{df}{dr}\cos\theta \right) - \big(\nabla^2\big(f(r)\big) - 1\big)\big(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\big) \right)$$

Some details.

$$v_r = u\cos\theta\left(1 - \frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r \to \infty) = \mathbf{u}$: $\quad \Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R) = 0$ solve for C_2, C_4

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

Assume that the velocity achieves steady flue far from the sphere and is zero on the sphere boundary.

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

$$\Rightarrow p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

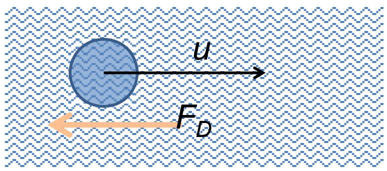
Finding all the constants and solving for the pressure .

$$p = p_0 - \eta u \cos\theta \left(\frac{3R}{2r^2} \right)$$

Corresponds to:

$$F_D \cos\theta = \left(p(R) - p_0 \right) 4\pi R^2$$

$$\Rightarrow \mathrm{F}_D = -\eta u \left(6\pi R \right)$$



Deducing the drag force from the solution to the differential equation.