PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF online

Plan for Lecture 34: Chap. 13 in F & W

Physics of elastic continua –

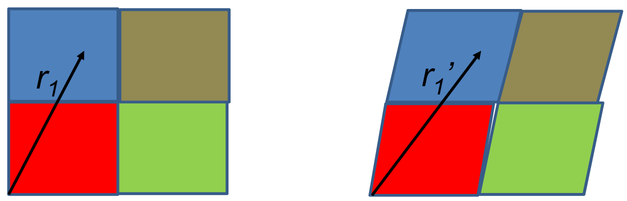
1. Stress and strain
2. Waves in elastic media

In this lecture we will explore some of the concepts of continuum elasticity discussed in Chapter 13 of your textbook. This is an old field and many overlapping notations.

27	Mon, 10/26/2020	Chap. 9	Mechanics of 3 dimensional fluids	#18	10/30/2020
28	Wed, 10/28/2020	Chap. 9	Mechanics of 3 dimensional fluids		
29	Fri, 10/30/2020	Chap. 9	Linearized hydrodynamics equations	#19	11/02/2020
30	Mon, 11/02/2020	Chap. 9	Linear sound waves	#20	11/04/2020
31	Wed, 11/04/2020	Chap. 9	Linear sound waves	Project topic	11/06/2020
32	Fri, 11/06/2020	Chap. 9	Sound sources and scattering; Non linear effects		
33	Mon, 11/09/2020	Chap. 9	Non linear effects in sound waves and shocks	#21	11/11/2020
34	Wed, 11/11/2020	Chap. 10	Surface waves in fluids	#22	11/16/2020
35	Fri, 11/13/2020	Chap. 10	Surface waves in fluids; soliton solutions		
36	Mon, 11/16/2020	Chap. 11	Heat conduction		
37	Wed, 11/18/2020	Chap. 12	Viscous effects		
38	Fri, 11/20/2020	Chap. 13	Elasticity		
39	Mon, 11/23/2020		Review		
	Wed, 11/25/2020		Thanksgiving Holidaya		
	Fri, 11/27/2020		Thanksgiving Holidaya		
40	Mon, 11/30/2020		Review		
	Wed, 12/02/2020		Presentations I		
	Fri, 12/04/2020		Presentations II		

Schedule.

Brief introduction to elastic continua



reference   deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' \approx \mathbf{r}_2 - \mathbf{r}_1 + \Big((\mathbf{r}_2 - \mathbf{r}_1) \cdot \nabla \Big) \mathbf{u}(\mathbf{r}_1) + \ldots$$

Visualization and quantification of deformation in terms of continuous deviation u(r).

Brief introduction to elastic continua -- continued

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \quad \epsilon_{ij} \quad + \quad O_{ij}$$

elastic strain tensor

rotation of material

Analyzing continuous deformation and distinguishing it from rotational effects.

Brief introduction to elastic continua -- continued

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

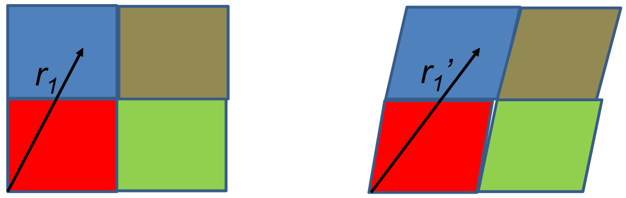
$$\equiv \epsilon_{ij} \qquad + \qquad \Omega_{ij}$$

$V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $\quad$ $V' = \mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}')$ $\quad$ $V' = V(1 + \nabla \cdot \mathbf{u}) = V\left(1 + \mathrm{Tr}(\epsilon)\right)$

$$\nabla \cdot \mathbf{u} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \mathrm{Tr}(\epsilon) = \frac{dV}{V} = -\frac{d\rho}{\rho}$$

In general we are interested in physical deformations and not rotation. The divergence of u is related to the fractional volume change or fractional density change.

Brief introduction to elastic continua -- continued



reference deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1) \cdot \nabla \right) \mathbf{u}(\mathbf{r}_1) + \ldots$$

$$x_{2i}' - x_{1i}' \approx x_{2i} - x_{1i} + \sum_{j=1}^{3} \epsilon_{ij} \left(x_{2j} - x_{1j} \right)$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

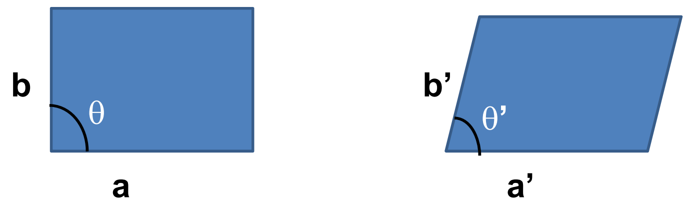
Effects of strain on a vector:

$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}} \right)$$

$$a' = \left| \mathbf{a}' \cdot \mathbf{a}' \right| \approx a\left(1 + \epsilon_{11} \right)$$

In addition to volumetric changes, there may be shape changes as well.

Deformation



$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$\mathbf{b}' = \mathbf{b} + b\left(\epsilon_{12}\hat{\mathbf{x}} + \epsilon_{22}\hat{\mathbf{y}} + \epsilon_{32}\hat{\mathbf{z}}\right)$$

$$\text{for } \mathbf{a}\cdot\mathbf{b} = 0 = ab\cos\theta \qquad \Rightarrow \theta = \frac{\pi}{2}$$

$$\mathbf{a}'\cdot\mathbf{b}' \approx ab\left(\epsilon_{21} + \epsilon_{12}\right) = 2ab\epsilon_{12} = ab\cos\theta'$$

$$\cos\theta' = \cos\left(\theta + (\theta' - \theta)\right) = \cos\theta\,\cos\left(\theta' - \theta\right) - \sin\theta\,\sin\left(\theta' - \theta\right)$$

$$\approx -\sin\theta\,\sin\left(\theta' - \theta\right) \approx -\left(\theta' - \theta\right)$$

$$\theta' \approx \theta - 2\epsilon_{12} = \frac{\pi}{2} - 2\epsilon_{12}$$

Off diagonal components of the strain tensor are related to angular deformations.

Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}} dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $F_x = -kx$:

Lame' coefficients : $$T_{ij} = -\lambda \delta_{ij} \nabla \cdot \mathbf{u} - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

or : $$T_{ij} = -\lambda \delta_{ij} \mathrm{Tr}(\epsilon) - 2\mu \epsilon_{ij}$$

Note: the elastic stress tensor is quite different from the stress tensor that we encountered in the hydrodynamic analysis!

11/20/2020 PHY 711 Fall 2020 -- Lecture 38 8

The generalization of Hooke's law; relating stress and strain.

Elastic stress tensor -- continued

$$T_{ij} = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$$

Note that: $\mathrm{Tr}(T) = -3\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{K \equiv \text{bulk modulus} = -V\left(\frac{\partial p}{\partial V}\right)}\mathrm{Tr}(\epsilon)$

Inverse Hooke's law: $\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij}Tr(T)\right)$

11/20/2020 PHY 711 Fall 2020 -- Lecture 38 9

Various relationships between stress and strain.

Stress tensor -- continued

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)} \delta_{ij} Tr(T) \right)$$

In terms of bulk modulus: $K = \lambda + \frac{2}{3}\mu$

$$\lambda = K - \frac{2}{3}\mu$$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij} Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3} \delta_{ij} Tr(T) \right)$$

Various parameters to describe elasticity.

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)} \;\delta_{ij}Tr(T) \right) = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\;\delta_{ij}Tr(T) \right)$$

Example -- hydrostatic pressure: $T_{ij} = \delta_{ij}dp$

$$\mathrm{Tr}(T) = 3dp$$

$$\epsilon_{ij} = -\frac{dp}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} \equiv \; -\frac{dp}{3K}\delta_{ij}$$

Note that: $\mathrm{Tr}(\epsilon) = \frac{dV}{V} = -\frac{dp}{K}$

$$\Rightarrow K = -V\frac{\partial p}{\partial V}$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)} \delta_{ij} Tr(T) \right) = -\frac{1}{9K}\delta_{ij} Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3} \delta_{ij} Tr(T) \right)$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

$\epsilon_{zz} = -\frac{1}{E} T_{zz}$ in terms of Young's modulus

$$E = \frac{9K\mu}{3K + \mu}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\left(\frac{1}{9K} - \frac{1}{6\mu} \right) dp$$

Poisson ratio: $\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = -\frac{\epsilon_{yy}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$

Uniaxial case and Young's modulus.

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\ \delta_{ij}Tr(T)\right)$$

Shear modulus

$$T_{ij} = \begin{cases} -f & \text{for } T_{xy} \text{ or } T_{yx} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{f}{2\mu}$$

Shere modulus.

Relationships between the elastic moduli:

Poisson's ratio:

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$$

Young's modulus:

$$E = \frac{9K\mu}{3K + \mu}$$

Shear modulus: μ

Relationships between elastic constants:

$$K = \frac{1}{3}\frac{E}{1-2\sigma}$$

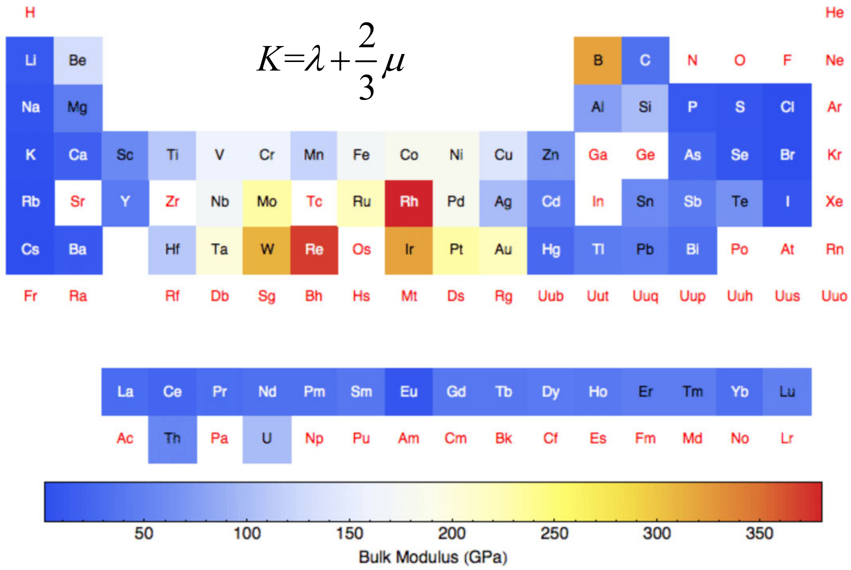
$$\mu = \frac{1}{2}\frac{E}{1+\sigma}$$

11/20/2020 PHY 711 Fall 2020 -- Lecture 38 14

Relationships between many parameters.

Values of bulk modulus K for elemental materials --

$$K=\lambda+\frac{2}{3}\mu$$

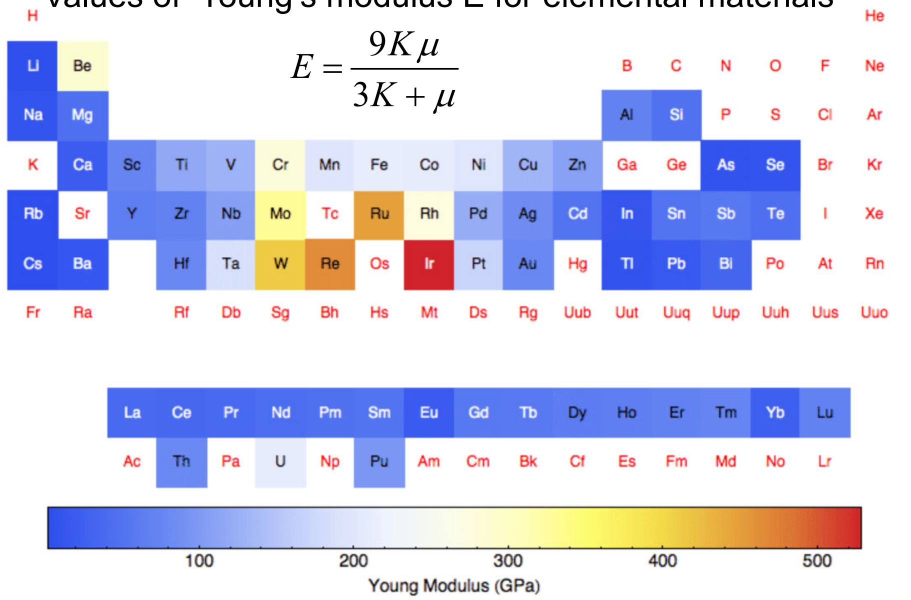


Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Elemental values of bulk modulus from Mathematica.

Values of Young's modulus E for elemental materials --

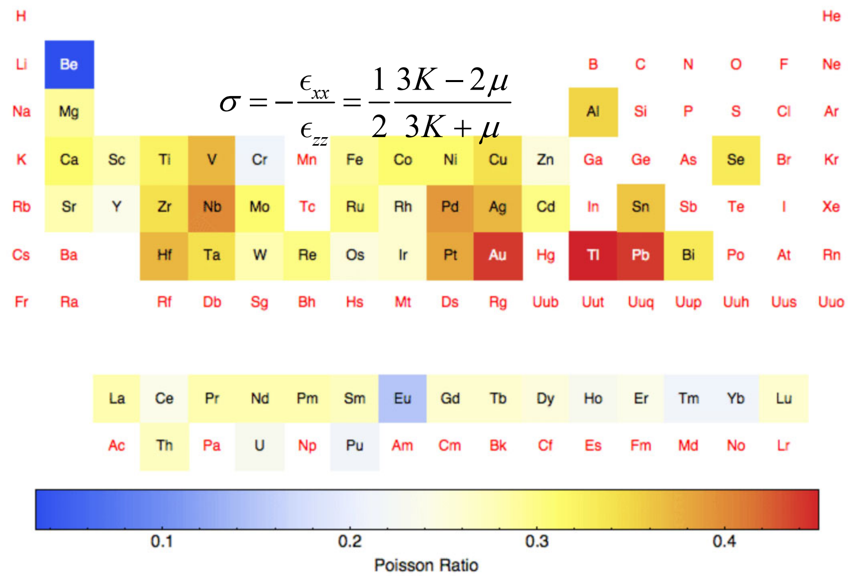
$$E = \frac{9K\mu}{3K+\mu}$$



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Values of Poisson ratio σ for elemental materials --

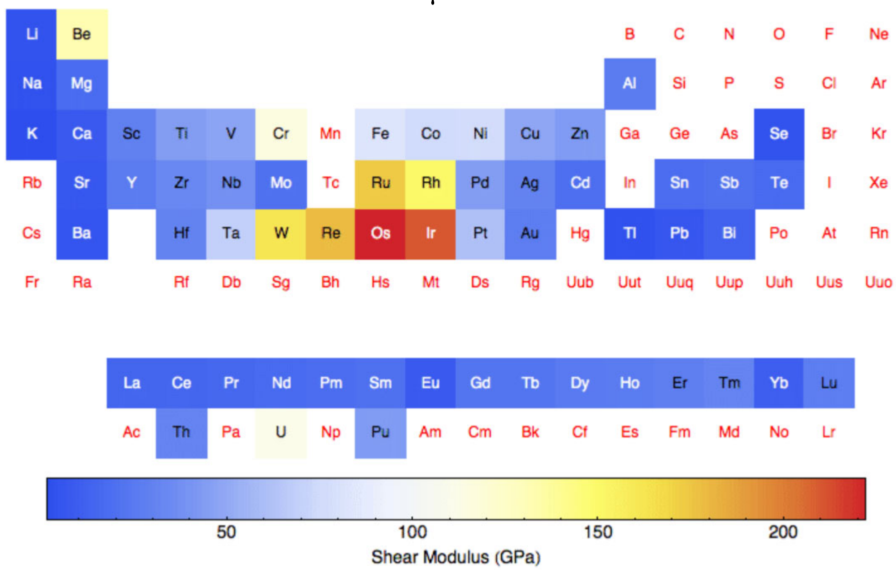
$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$$



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

Poisson ratio also from Mathematica.

Values of shear modulus μ for elemental materials --



Up to date, curated data provided by *Mathematica*'s ElementData function from Wolfram Research, Inc.

11/20/2020 PHY 711 Fall 2020 -- Lecture 38 18

Shear modulus.

Summary -- Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}} dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $\mathrm{F}_x = -kx$:

Lame' coefficients : $$T_{ij} = -\lambda \delta_{ij} \nabla \cdot \mathbf{u} - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -\lambda \delta_{ij} \mathrm{Tr}(\epsilon) - 2\mu \epsilon_{ij}$$

material-dependent empirical parameters

Bulk modulus: $K = \lambda + \frac{2}{3}\mu$

Young's modulus: $E = \frac{9K\mu}{3K + \mu}$

Poisson ratio: $\sigma = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$

Shear modulus: μ

Summary of relationships.

Stress   Strain

Hooke's Law: $T_{ij} = -K\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}(\epsilon)\right)$

Inverse Hooke's Law: $\epsilon_{ij} = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$

Elastic force: $F_i^{\text{elastic}} = \sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}$

Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x\;\mathbf{F}^{\text{elastic}}\cdot\delta\mathbf{u}$$

$$= \int_V d^3x\;\sum_{ij=1}^{3}\frac{\partial T_{ij}}{\partial x_j}\delta u_i$$

$$= \sum_{ij=1}^{3}\int_A dA_j\;T_{ij}\delta u_i - \sum_{ij=1}^{3}\int_V d^3x T_{ij}\frac{\partial \delta u_i}{\partial x_j}$$

More summary.

Elastic work due to distortion $\delta\mathbf{u}$

$$\delta W^{\text{elastic}} = \int_V d^3x \; \mathbf{F}^{\text{elastic}} \cdot \delta\mathbf{u}$$

$$= \sum_{ij=1}^{3} \int_A dA_j \; T_{ij}\delta u_i - \sum_{ij=1}^{3} \int_V d^3x T_{ij} \frac{\partial \delta u_i}{\partial x_j}$$

$$= \sum_{ij=1}^{3} \int_A dA_j \; T_{ij}\delta u_i - \sum_{ij=1}^{3} \int_V d^3x T_{ij}\epsilon_{ij}$$

surface contribution | bulk contribution

For large samples: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3} \int_V d^3x T_{ij}\epsilon_{ij}$

Work involved with distortion.

Bulk elastic energy: $\delta W^{\text{elastic}} = -\sum_{ij=1}^{3}\int_V d^3x T_{ij}\epsilon_{ij}$

Integrating from 0 to final strain ϵ_{ij} :

$$\delta W^{\text{elastic}} = -\frac{1}{2}\sum_{ij=1}^{3} T_{ij}\epsilon_{ij}$$

Hooke's Law: $T_{ij} = -K\delta_{ij}\text{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\text{Tr}(\epsilon)\right)$

$$\delta W^{\text{elastic}} = \frac{1}{2}K(\text{Tr }\epsilon)^2 + \mu\sum_{ij=1}^{3}\left(\epsilon_{ij} - \frac{\delta_{ij}}{3}\text{Tr }\epsilon\right)^2$$

$$= \frac{1}{2}\lambda(\text{Tr }\epsilon)^2 + \mu\sum_{ij=1}^{3}\epsilon_{ij}^2$$

22

Analysis of the elastic work using the Hooke’s law relationships.

Note that the two relations:

$$\delta W^{\text{elastic}} = \frac{1}{2}K\left(\text{Tr }\epsilon\right)^2 + \mu\sum_{ij=1}^{3}\left(\epsilon_{ij} - \frac{\delta_{ij}}{3}\text{Tr }\epsilon\right)^2$$

$$T_{ij} = -K\delta_{ij}\text{Tr}(\epsilon) - 2\mu\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\text{Tr}(\epsilon)\right)$$

Ensure that: $\dfrac{\partial \delta W^{\text{elastic}}}{\partial \epsilon_{ij}} = -T_{ij}$

More details.

Dynamical equations of motion

Recall Newton's second law for continuum system with density ρ, velocity components v_k and stress tensor T_{ij} :

$$\frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} + \rho f_k$$

external force density

For our elastic medium: ρ does not vary in time

Velocity related to displacement: $v_k = \frac{\partial u_k}{\partial t}$

Hooke's law:

$$T_{kl} = -K\delta_{kl}\mathrm{Tr}\left(\epsilon\right) - 2\mu\left(\epsilon_{kl} - \frac{1}{3}\delta_{kl}\mathrm{Tr}\left(\epsilon\right)\right)$$

$$= -K\delta_{kl}\left(\nabla\cdot\mathbf{u}\right) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) - \frac{1}{3}\delta_{kl}\left(\nabla\cdot\mathbf{u}\right)\right)$$

Analyzing the time dependence of the elastic response in terms of Newton's laws. While these equations look like the analysis in hydrodynamics, the stress tensor is quite different and nonlinear effects are more accurately ignored.

For: $T_{kl} = -K\delta_{kl}\left(\nabla\cdot\mathbf{u}\right) - 2\mu\left(\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)-\frac{1}{3}\delta_{kl}\left(\nabla\cdot\mathbf{u}\right)\right)$

$$\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} = -\left(K-\frac{2}{3}\mu\right)\sum_{l=1}^{3}\left(\delta_{kl}\frac{\partial\left(\nabla\cdot\mathbf{u}\right)}{\partial x_l}\right) - \mu\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} - \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$= -\left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} - \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l}$$

$$\frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3}\frac{\partial T_{kl}}{\partial x_l} + \rho f_k \qquad v_k = \frac{\partial u_k}{\partial t}$$

$$\rho\frac{\partial^2 u_k}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_k\partial x_l} + \mu\sum_{l=1}^{3}\frac{\partial^2 u_k}{\partial^2 x_l} + \rho f_k$$

Vector form: $\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \left(K+\frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right) + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}$

Analyzing the equations of motion to linear order.

Dynamical equations of elastic continuum

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left(K + \frac{1}{3}\mu \right) \nabla \left(\nabla \cdot \mathbf{u} \right) + \rho \mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla \times \mathbf{u}_l = 0$ and $\nabla \cdot \mathbf{u}_t = 0$

$$c_l = \left(\frac{K + \frac{4}{3}\mu}{\rho} \right)^{1/2} \quad \text{and} \quad c_t = \left(\frac{\mu}{\rho} \right)^{1/2}$$

Considering longitudinal and transverse wave motion in elastic media.

Typical velocities of longitudinal sound waves

http://www.engineeringtoolbox.com/sound-speed-solids-d_713.html

air: c_l=343 m/s
water: c_l=1433 m/s

Material	c_l (m/s)
Aluminum, shear - longitudinal wave	3100 - 6400
Beryllium	12890
Brass	3475
Brick	4176
Concrete	3200 - 3600
Copper	4600
Cork	366 - 518
Diamond	12000
Glass	3962
Glass, Pyrex	5640
Gold	3240
Granite	5950
Hardwood	3962
Iron	5130
Lead	1960 - 2160
Lucite	2680
Rubber, butyl	1830
Rubber	40 - 150
Silver	3650
Steel	6100
Steel, stainless	5790
Titanium	6070
Wood (hard)	3960
Wood	3300 - 3600

Longitudinal wave speeds in various media.

from:
https://pangea.stanford.edu/courses/gp262/Notes/5.Elasticity.pdf

Mineral	Density	Young's Modulus	Bulk Modulus	Shear Modulus	Vp	Vs	Poisson's Ratio
Quartz	2.6500	95.756	36.600	45.000	6.0376	4.1208	0.063953
Calcite	2.7100	84.293	76.800	32.000	6.6395	3.4363	0.31707
Dolomite	2.8700	116.57	94.900	45.000	7.3465	3.9597	0.29527
Clay (kaolinite)	1.5800	3.2034	1.5000	1.4000	1.4597	0.94132	0.14407
Muscovite	2.7900	100.84	61.500	41.100	6.4563	3.8381	0.22673
Feldspar (Albite)	2.6300	69.010	75.600	25.600	6.4594	3.1199	0.34786
Halite	2.1600	37.242	24.800	14.900	4.5474	2.6264	0.24972
Anhydrite	2.9800	74.431	56.100	29.100	5.6432	3.1249	0.27888
Pyrite	4.9300	305.85	147.40	132.50	8.1076	5.1842	0.15417
Siderite	3.9600	134.51	123.70	51.000	6.9576	3.5887	0.31876
gas	0.00065000	0.0000	0.00013000	0.0000	0.44721	0.0000	0.50000
water	1.0000	0.0000	2.2500	0.0000	1.5000	0.0000	0.50000
oil	0.80000	0.0000	1.0200	0.0000	1.1292	0.0000	0.50000

C.1

densities in g/cm³
moduli in GPa
velocities in km/s

More measured values of elasticity values for various materials.

Dynamical equations of elastic continuum

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left(K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u}) + \rho \mathbf{f}$$

In absense of external force:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left(K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u})$$

Suppose: $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla \times \mathbf{u}_l = 0$ and $\nabla \cdot \mathbf{u}_t = 0$

$$\rho \frac{\partial^2 (\mathbf{u}_l + \mathbf{u}_t)}{\partial t^2} = \mu \nabla^2 (\mathbf{u}_l + \mathbf{u}_t) + \left(K + \frac{1}{3}\mu \right) \nabla (\nabla \cdot \mathbf{u}_l)$$

Some more details.

Dynamical equations of elastic continuum

Transverse component:

$$\rho \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \mu \nabla^2 \mathbf{u}_t \qquad \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \mathbf{u}_t \equiv c_t^2 \nabla^2 \mathbf{u}_t$$

Transverse wave velocity: $\mathrm{c}_t = \sqrt{\frac{\mu}{\rho}}$

Longitudinal component: $\rho \frac{\partial^2 \mathbf{u}_l}{\partial t^2} = \mu \nabla^2 \mathbf{u}_l + \left(K + \frac{1}{3}\mu \right) \nabla \left(\nabla \cdot \mathbf{u}_l \right)$

Note that the longitudinal wave has its displacement along its propagation direction x_p, so that $\mathbf{u}_l = \mathbf{u}_l\left(x_p \right) \equiv u_l(x_p)\hat{\mathbf{x}}_p$

$$\Rightarrow \rho \frac{\partial^2 u_l}{\partial t^2} = \mu \frac{\partial^2 u_l}{\partial x_p^2} + \left(K + \frac{1}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2} = \left(K + \frac{4}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2}$$

More details.

Dynamical equations of elastic continuum

Longitudinal component -- continued:

$$\rho \frac{\partial^2 u_l}{\partial t^2} = \left(K + \frac{4}{3}\mu \right) \frac{\partial^2 u_l}{\partial x_p^2}$$

$$\frac{\partial^2 u_l}{\partial t^2} = \left(\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho} \right) \frac{\partial^2 u_l}{\partial x_p^2} \equiv c_l^2 \frac{\partial^2 u_l}{\partial x_p^2}$$

Longitudinal wave velocity: $\mathrm{c}_l = \sqrt{\frac{K}{\rho} + \frac{4}{3}\frac{\mu}{\rho}}$

Transverse component:

$$\rho \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \mu \nabla^2 \mathbf{u}_t \qquad \frac{\partial^2 \mathbf{u}_t}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \mathbf{u}_t \equiv c_t^2 \nabla^2 \mathbf{u}_t$$

Transverse wave velocity: $\mathrm{c}_t = \sqrt{\frac{\mu}{\rho}}$

Summary.

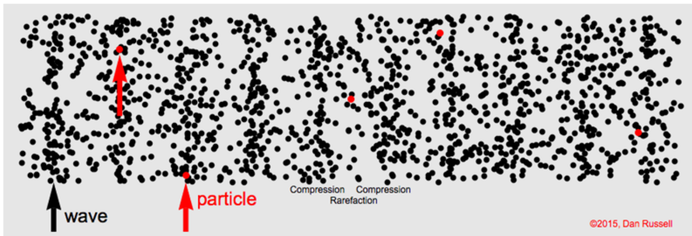
Some values:

Substance	Density (g/cm^3)	V_l (m/s)	V_s (m/s)
Metals			
Aluminum, rolled	2.7	6420	3040
Berylium	1.87	12890	8880
Brass (70 Cu, 30 Zn)	8.6	4700	2110
Copper, annealed	8.93	4760	2325
Copper, rolled	8.93	5010	2270
Gold, hard-drawn	19.7	3240	1200
Iron, Armco	7.85	5960	3240
Lead, annealed	11.4	2160	700
Lead, rolled	11.4	1960	690
Molybdenum	10.1	6250	3350
Monel metal	8.90	5350	2720
Nickel (unmagnetized)	8.85	5480	2990
Nickel	8.9	6040	3000
Platinum	21.4	3260	1730
Silver	10.4	3650	1610
Steel, mild	7.85	5960	3235
Steel, 347 Stainless	7.9	5790	3100
Tin, rolled	7.3	3320	1670
Titaninum	4.5	6070	3125
Tungsten, annealed	19.3	5220	2890
Tungsten Carbide	13.8	6655	3980
Zinc, rolled	7.1	4210	2440
Various			
Fused silica	2.2	5968	3764
Glass, pyrex	2.32	5640	3280
Glass, heavy silicate flint	3.88	3980	2380
Lucite	1.18	2680	1100
Nylon 6-6	1.11	2620	1070
Polyethylene	0.90	1950	540
Polystyrene	1.06	2350	1120

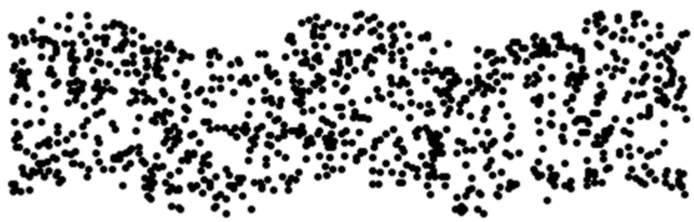
Sound velocities in various materials.

Animations from website:
https://www.acs.psu.edu/drussell/demos/waves/wavemotion.html

Longitudinal wave (p or “primary”)

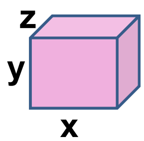


Transverse wave (s or “secondary”)



Visualization of longitudinal and transverse waves.

Elasticity in solids



We imagine that near equilibrium, the solid can be described in terms of a potential function $\phi(\{\mathbf{R}\})$ where $\{\mathbf{R}\}$ represents the positions of each atom:

$$E = \sum_{\mathbf{R}} \phi(\{\mathbf{R}\}) + \frac{1}{2}\sum_{\mathbf{RR}'} (\mathbf{u}(\mathbf{R}) - \mathbf{u}(\mathbf{R}')) \cdot \nabla \phi(\{\mathbf{R}\}) + \frac{1}{4}\sum_{\mathbf{RR}'} ((\mathbf{u}(\mathbf{R}) - \mathbf{u}(\mathbf{R}')) \cdot \nabla)^2 \phi(\{\mathbf{R}\}) + \ldots$$

vanishes at equilibrium

$$\delta W^{\text{elastic}} = \frac{1}{4}\sum_{\substack{\mathbf{RR}' \\ ij}} (u_i(\mathbf{R}) - u_i(\mathbf{R}')) \frac{\partial^2 \phi(\{\mathbf{R}\})}{\partial u_i \partial u_j} (u_j(\mathbf{R}) - u_j(\mathbf{R}'))$$

Comment on elasticity in solids.

$$\delta W^{\text{elastic}} = \frac{1}{4}\sum_{\substack{\mathbf{RR'}\\ ij}} \left(u_i(\mathbf{R}) - u_i(\mathbf{R'})\right) \frac{\partial^2 \phi(\{\mathbf{R}\})}{\partial u_i \partial u_j} \left(u_j(\mathbf{R}) - u_j(\mathbf{R'})\right)$$

Note that $\mathbf{u}(\mathbf{R'}) \approx \mathbf{u}(\mathbf{R'}) + (\mathbf{R'} - \mathbf{R}) \cdot \nabla \mathbf{u}(\mathbf{R})$

In terms of strain coeffients ϵ_{ij} :

$$\delta W^{\text{elastic}} = \frac{1}{2}\sum_{ijkl} \epsilon_{ij} c_{ijkl} \epsilon_{kl}$$

where coeffients c_{ijkl} are composed of permutations of $R_i \frac{\partial^2 \phi}{\partial u_j \partial u_k} R_l$

For the most general case c_{ijkl} have 21 distrinct terms, for a cube there are only 3 unique terms.

More details.

Simplified notation:

$xx \to 1$

$yy \to 2$

$zz \to 3$

$yz \to 4$

$zx \to 5$

$xy \to 6$

For cubic crystals, the unique coefficients are:

$C_{11} = c_{xxxx}$ $\quad C_{12} = c_{xxyy}$ $\quad C_{44} = c_{yzyz}$

Some typical values (Ref. Ashcroft and Mermin (1976))

	C_{11} (GPa)	C_{12}(GPa)	C_{44}(GPa)
Na	7.0	6.1	4.5
Al	107	61	28
Fe	234	136	118
Si	166	64	80
NaCl	48.7	12.4	12.6

Examples of stress coefficients for some materials.