PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Plan for Lecture 3: -- Chap. 1 of F&W

Scattering theory

1. **Background and motivation**
2. **Center of mass reference fame and its relationship to "laboratory" reference frame**
3. **Next time -- Analytical evaluation of the differential scattering cross section in general and for Rutherford scattering**

This lecture will continue the discussion of scattering theory. After discussing a little bit more of the background and motivation, we will jump into the geometry of the process. Typical "lab" experiments are hard to analyze directly. However, scattering in the center of mass frame of reference is much more amenable to detailed analysis. In this lecture we will discuss how we can relate the results in the two frames of reference. Next time, we will analyze the center of mass scattering for a few examples.

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	OPL 103	http://www.wfu.edu/~natalie/f20phy711/

Instructor: Natalie Holzwarth	**Office:** 300 OPL	**e-mail:** natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 2	Non-inertial coordinate systems		
6	Mon, 9/07/2020	Chap. 3	Calculus of Variation		

This is the schedule posted on the webpage.

PHY 711 -- Assignment #3

Aug. 31, 2020

Read Chapter 1 in **Fetter & Walecka**.

1. In Lecture 3, we derived equations relating the laboratory scattering angle to the scattering angle in the center of mass reference frame. We also worked out the relationship between the differential scattering cross sections in the laboratory and center of mass frames. After you have convinced yourselves of the validity of those derivations, evaluate both the lab and center of mass scattering angles and the corresponding cross section factors for the following mass ratios
    a. m_1/m_2=1
    b. m_1/m_2=1/10
    c. m_1/m_2=10/1

This is the homework problem posted for this lecture.

Scattering theory:

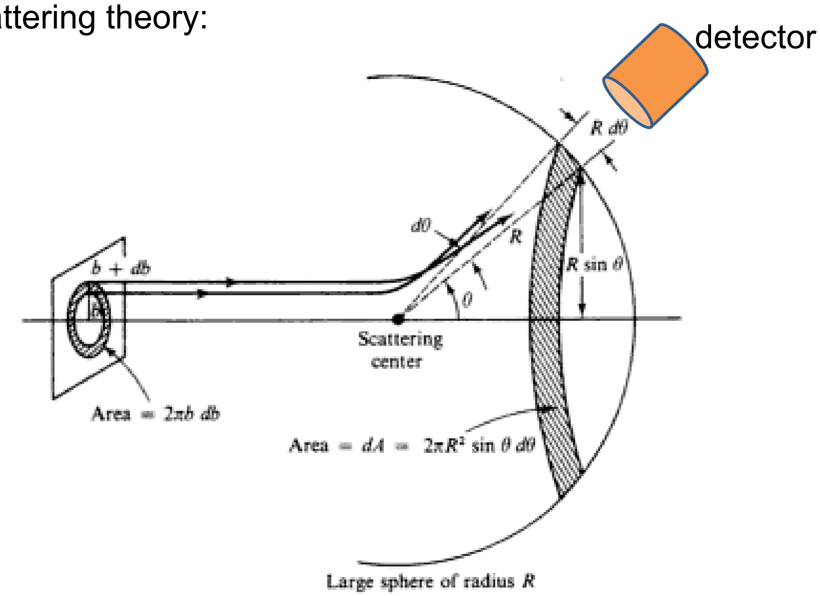


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

8/31/2020 PHY 711 Fall 2020 -- Lecture 3 4

This is the image of the ideal scattering geometry that is in your textbook. Later in the lecture we will reference this diagram as the “lab” frame of reference. The target particle (#2) is assumed to be initially at rest, while the scattering particle (#1) is moving at constant velocity toward the target (when it is far from the target). As the scattering particle approaches the target, it is deflected as shown and ends up in the detector place at the scattering angle theta.

Can you think of examples of such an experimental setup?

Other experimental designs –

At CERN https://home.cern/science/experiments/totem the study of highly energetic proton-proton scattering is designed in the center of mass frame of reference by accelerating two proton beams focused to collide head on in the Large Hadron Collider LHC facility.

Figure from CERN website

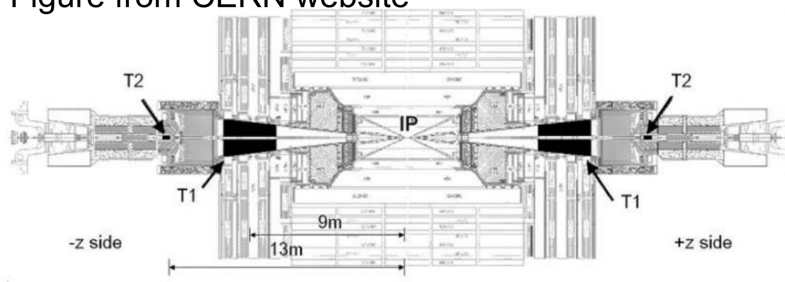


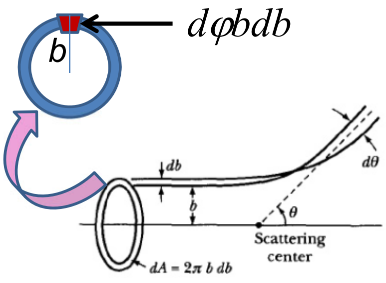
Figure 1.17: View of the inelastic forward trackers T1 and T2 inside the CMS detector.

Think about some situations you know about where such a laboratory experiment has been carried out. Of course, the stationary target arrangement is not always the case. This slide mentions a particular experiment at CERN where the experiment is actually designed to be in the center of mass frame. This is a special arrangement and the physics under consideration goes far beyond classical mechanics. In fact, it is a highly inelastic collision.

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$

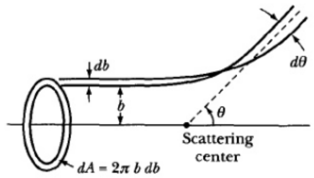


$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

For the normal (stationary target) setup, it is very useful to standardize the measurement of scattering in terms of a "cross section". Furthermore, for classical mechanics, where we can analyze the particle trajectories in detail, we can calculate the cross section in terms of particle parameters. While our analysis will consider a single particle in the vicinity of the target, actual experiments are performed with many such particles in a beam. All beam particles are initially moving toward the target with a given initial kinetic energy. Within the beam of particles, there will be a distribution of "impact parameters" b defined in the diagram. As we will see, the impact parameter of the particple will determine its scattering angle theta. From geometry, the function b(theta) determines the differential cross section from the expression given on this slide.

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



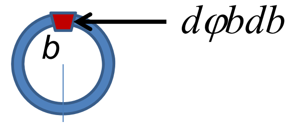


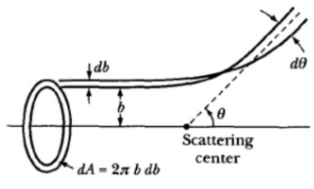
Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

This slide shows a little more detail of the same ideas.

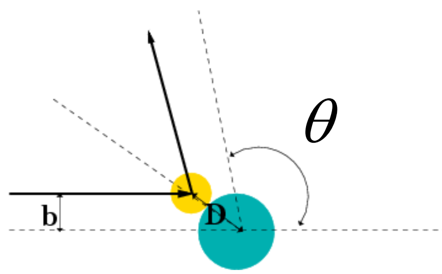
Simple example – collision of hard spheres



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

$$b(\theta) = ?$$

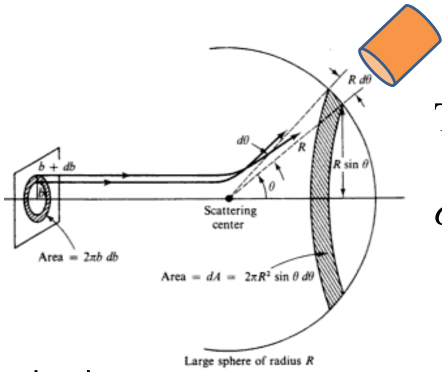


$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

Here we show the example for the case that the scattering particle and the target interact as ideal hard spheres, assuming the target particle is stationary and infinitely massive. In this case the cross section depends on the distance of closest approach D.

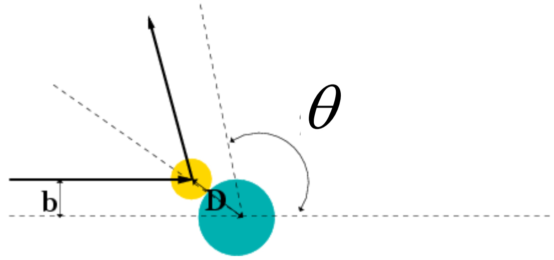
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:



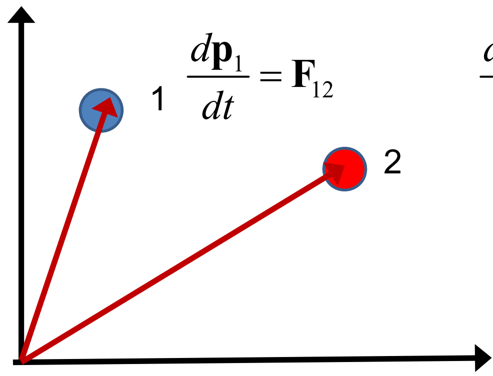
$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

Additional comments on the differential and total cross sections.

Now consider the more general case of particle interactions and the corresponding scattering analysis.

Relationship of scattering cross-section to particle interactions --
Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \quad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

This is the general diagram we showed last time. We will assume that there are no additional forces acting on the system and that the forces are entirely conservative so that energy E (sum of kinetic and potential) is conserved

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \leq a \\ 0 & r > a \end{cases}$

Coulomb or gravitational: $V(r) = \frac{K}{r}$

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

Scattering theory can help us analyze the interaction potential *V(r).* First we need to simply the number of variables.

8/31/2020 PHY 711 Fall 2020 -- Lecture 3 12

We will assume that the potential depends only on the particle distance and has no angular dependence.

Relationship between center of mass and laboratory frames of reference. At and time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \left(m_1 + m_2\right)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \left(m_1 + m_2\right)\dot{\mathbf{R}}_{CM} = \left(m_1 + m_2\right)\mathbf{V}_{CM}$$

Note that $\dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{dt}$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

$$= \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

8/31/2020 PHY 711 Fall 2020 -- Lecture 3 13

It is convenient to transform our coordinate system from particle 1 and particle 2 to instead the center of mass coordinate and the relative coordinate.

Classical mechanics of a conservative 2-particle system -- continued

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r_{12})$

Relative angular momentum is also conserved:

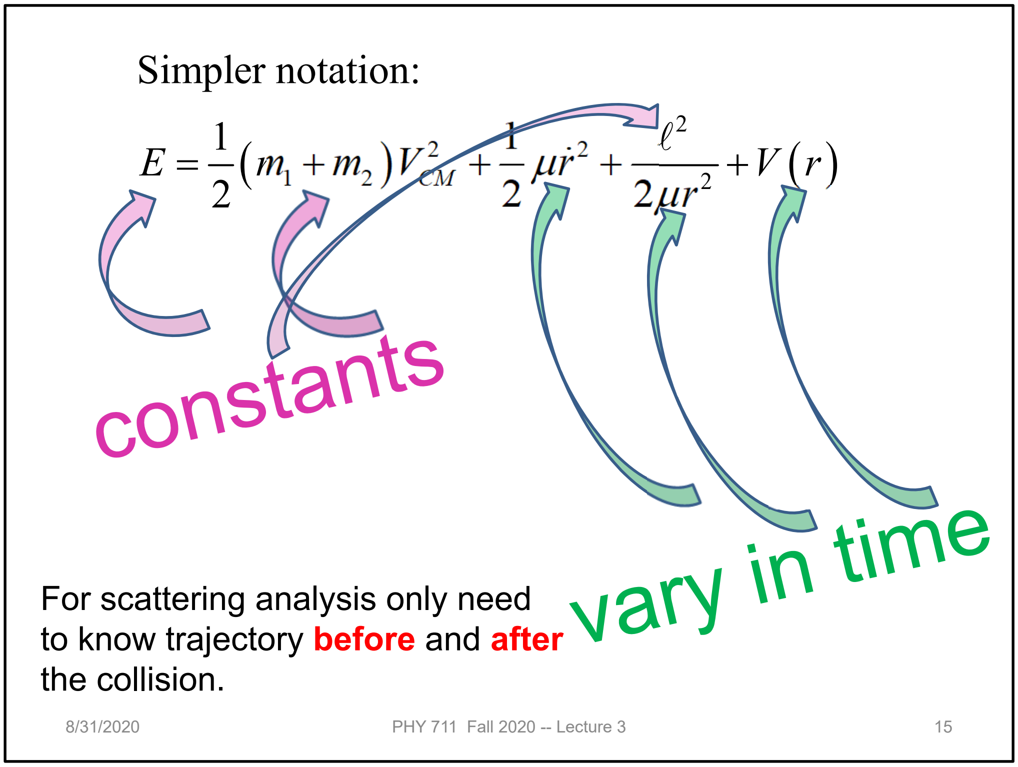
$$\mathbf{L}_{12} \equiv \mathbf{r}_{12} \times \mu\mathbf{v}_{12}$$

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu v_{12}^2 + \frac{L_{12}^2}{2\mu r_{12}^2} + V(r_{12})$$

Simpler notation:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

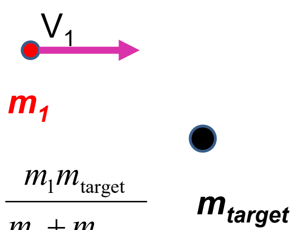
If there are no external forces on this system, the kinetic energy of the center of mass is constant. Additionally, for the central interaction potential , the relative angular momentum is also conserved. The last equation shows that the important physics of the system is determined by the total energy of the relative system.



Here we highlight the last result. The green arrows point to the relative coordinate terms that will be the focus of our analysis. For the scattering process, only the before and after processes directly enter the analysis. Of course the after parameters depend on the before parameters and the intermediate trajectory.

Note: The following analysis will be carried out in the center of mass frame of reference.

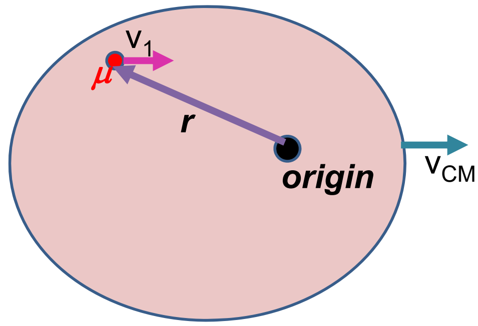
In laboratory frame:



$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = \left| \mathbf{r} \times \mu \mathbf{v}_1 \right|$$

In center-of-mass frame:



Also note: We are assuming that the interaction between particle and target *V(r)* conserves energy and angular momentum.

In the next several slides we will consider the scattering geometry and the relationships between the two coordinate systems.

It is often convenient to analyze the scattering cross section in the center of mass reference frame.

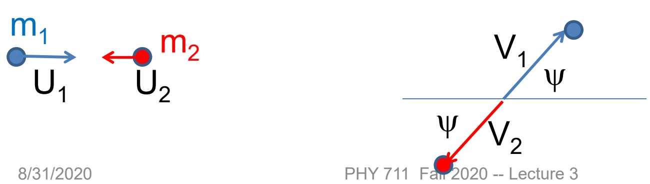
Relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before | After

Center of mass reference frame:

Before | After



Here we define the notation, keep theta as the scattering angle measured in the lab frame.

Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1+m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

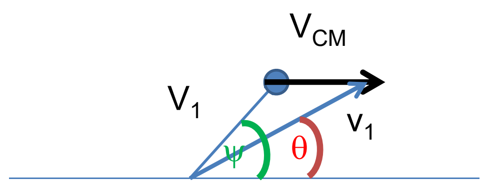
$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1+m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

u and U denote before the collision and v and V denote after the collision. Lower case references the lab frame and upper case references the center of mass frame.

Relationship between center of mass and laboratory frames of reference for the scattering particle 1



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

For elastic scattering

Focusing on the variables describing particle 1 after the collision.

Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2 + \tfrac{1}{2}m_2U_2^2 + \tfrac{1}{2}(m_1+m_2)V_{CM}^2$$
$$= \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 + \tfrac{1}{2}(m_1+m_2)V_{CM}^2$$

Also note:

$$m_1\mathbf{U}_1 + m_2\mathbf{U}_2 = 0 \qquad m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = 0$$

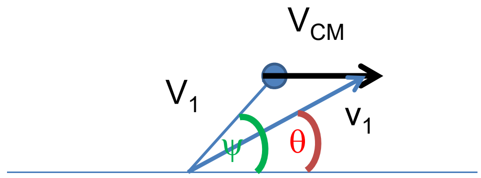
$$\mathbf{U}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM} \qquad \mathbf{U}_2 = -\mathbf{V}_{CM}$$

$$\Rightarrow |\mathbf{U_1}| = |\mathbf{V_1}| \quad \text{and} \quad |\mathbf{U_2}| = |\mathbf{V_2}| = |\mathbf{V_{CM}}|$$

Also note that : $m_1|\mathbf{U_1}| = m_2|\mathbf{U_2}|$

So that : $V_{CM}/V_1 = V_{CM}/U_1 = m_1/m_2$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM} / V_1} = \frac{\sin\psi}{\cos\psi + m_1 / m_2}$$

Also: $$\cos\theta = \frac{\cos\psi + m_1 / m_2}{\sqrt{1 + 2m_1 / m_2 \cos\psi + \left(m_1 / m_2\right)^2}}$$

After some algebra.

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}} = \left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right| = \left|\frac{d\cos\psi}{d\cos\theta}\right|$$

Using:

$$\cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1+2(m_1/m_2)\cos\psi + (m_1/m_2)^2}}$$

$$\left|\frac{d\cos\theta}{d\cos\psi}\right| = \frac{(m_1/m_2)\cos\psi + 1}{\left(1+2(m_1/m_2)\cos\psi + (m_1/m_2)^2\right)^{3/2}}$$

More steps.

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

$$\text{where:} \quad \tan\theta = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

Summary of results

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

Example: suppose $m_1 = m_2$

In this case: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+1} \quad \Rightarrow \theta=\dfrac{\psi}{2}$

note that $0\le\theta\le\dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\theta)}{d\Omega_{CM}}\right)\cdot 4\cos\theta$$

Summary --

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $\tan\theta = \dfrac{\sin\psi}{\cos\psi + m_1/m_2}$ For elastic scattering

Hard sphere example – continued

m_1=m_2

Center of mass frame

$$\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)=\frac{D^2}{4}$$

$$\int\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}d\Omega_{CM}=$$

$$\frac{D^2}{4}4\pi=\pi D^2$$

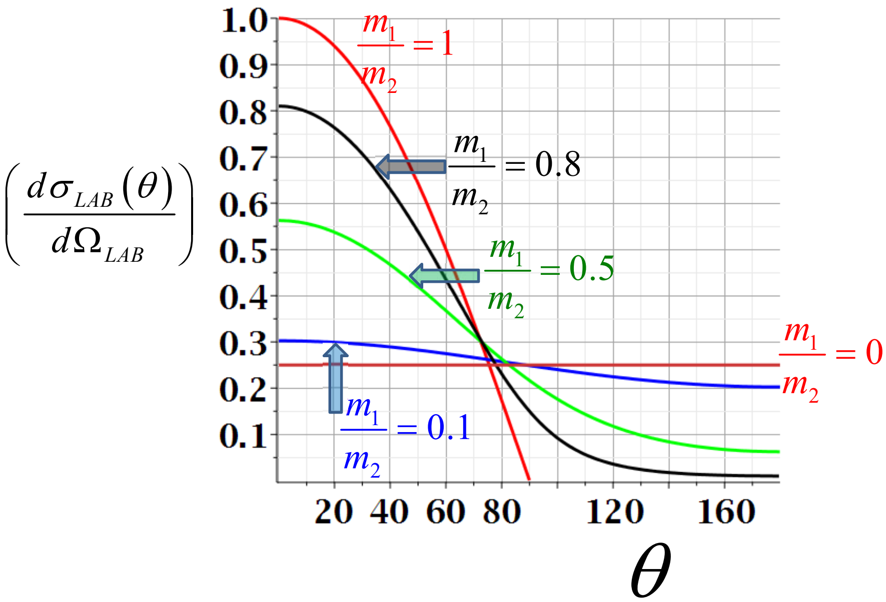
Lab frame

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=D^2\cos\theta \quad \theta=\frac{\psi}{2}$$

$$\int\frac{d\sigma_{lab}(\theta)}{d\Omega_{lab}}d\Omega_{lab}=$$

$$2\pi D^2\int_0^{\pi/2}\cos\theta\,\sin\theta d\theta=\pi D^2$$

Scattering cross section for hard sphere in lab frame for various mass ratios:



Plot of the lab cross sections for various mass ratios.

For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \theta(\psi)$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right) \frac{\left(1 + 2m_1/m_2 \cos\psi + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi + 1}$$

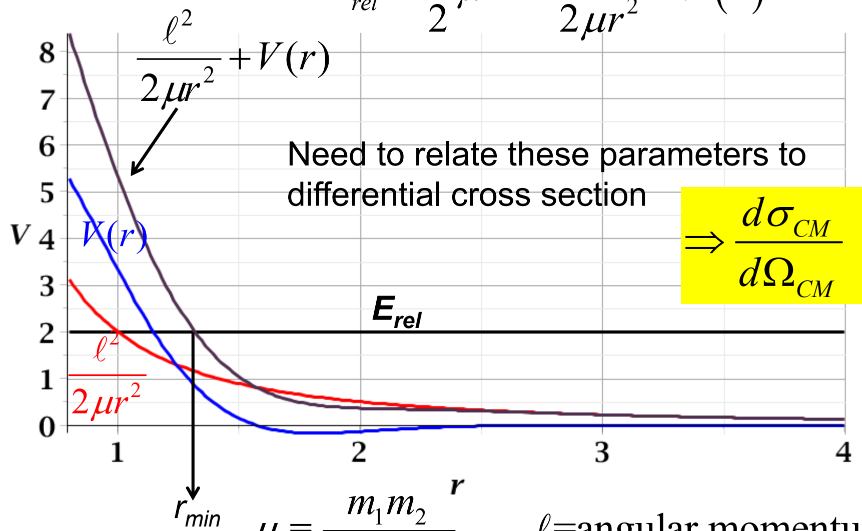
where: $\tan\theta = \dfrac{\sin\psi}{\cos\psi + m_1/m_2}$

Maple syntax:

```
> plot( { [psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
   = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
  sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
  thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
  orange])
```

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

Diagram for the relative coordinates

Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \leq D \\ 0 & r > D \end{cases}$ X

Coulomb or gravitational: $V(r) = \frac{K}{r}$ ←

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

We will consider this next time.