PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Discussion notes for Lecture 4

Scattering analysis in the center of mass frame.

1. **Summary of what we have learned so far**
2. **Analytical evaluation of the differential scattering cross section in general and for Rutherford scattering (in center of mass frame)**

Physics colloquium Thursday at 4 PM – https://www.physics.wfu.edu/

Announcements

Prof. Kandada's work featured in Emerging Investigators issue

Prof. Salsbury Named Scott Family Faculty Fellow

Mission

Nationally recognized for teaching excellence; internationally respected for research advances; a focused emphasis on interdisciplinary study and close student-faculty collaboration; committed to a diverse and inclusive environment.

Events

Sep 03 2020

Colloquium: "Welcome to the WFU Physics Department." – September 3, 2020 at 4 PM

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PROGRAM

The purpose of this first seminar is to help new, returning, and prospective students (including both undergraduate and graduate students), faculty, and staff to become acquainted with each other and with the Physics Department. We will meet online at 4:00 PM for presentations by some undergraduate students highlighting their summer research experiences, followed by general welcoming statements and departmental announcements. We are especially pleased to welcome quite a few new additions to the physics department.

Date

Sep 03 2020

Time

4:00 pm - 5:00 pm

CATEGORY

› Colloquium

Share this event

You will need a link to the video conference which you should receive if you are registered for PHY 601 or you can request to be on the mailing list -- contact Kittye McBride mcbridek@wfu.edu.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory		
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation		

Note that there is no new assigned homework for this lecture. Friday's lecture will review scattering theory and answer your remaining questions. After today's lecture and before 7 AM Friday, continue to send me your questions on this material and on Chapter 1 of F&W.

Your questions –
From Tim

1. So if you want your scattered particles to appear at a certain angle (theta), then you should play with the input parameter b until your particles are being picked up by your fixed detector? I guess the E*rel* could also be played with to vary K in the equation.

From Jeanette

1. In E = E_CM + E _rel, what does rel mean? Relative?
2. Slide 16 - E_rel only has one term, but in slide 14 it had 3 terms. What happened to the other 2 terms?

From Gao

1. For defined interparticle potential the trajectory of a scattering particle is defined. So what physical meanings is the differential cross section?

From Nick

1. What is meant by solid angle? .
2. The book and slides have slightly different formulations and I'm trying to rectify the two. Can you go over again how we get , what it represents, and why it's significant? In particular, why is from the slides and from the text?
3. Can you clarify the solid angle from lab vs. CM frames.
4. Where does the 4pi come from on lecture 3 extra, slide 35?
5. Just to clarify, \phi and \varphi are the same thing right?
6. The last line on slide 20 of lecture 4...can you clarify?

From Derek

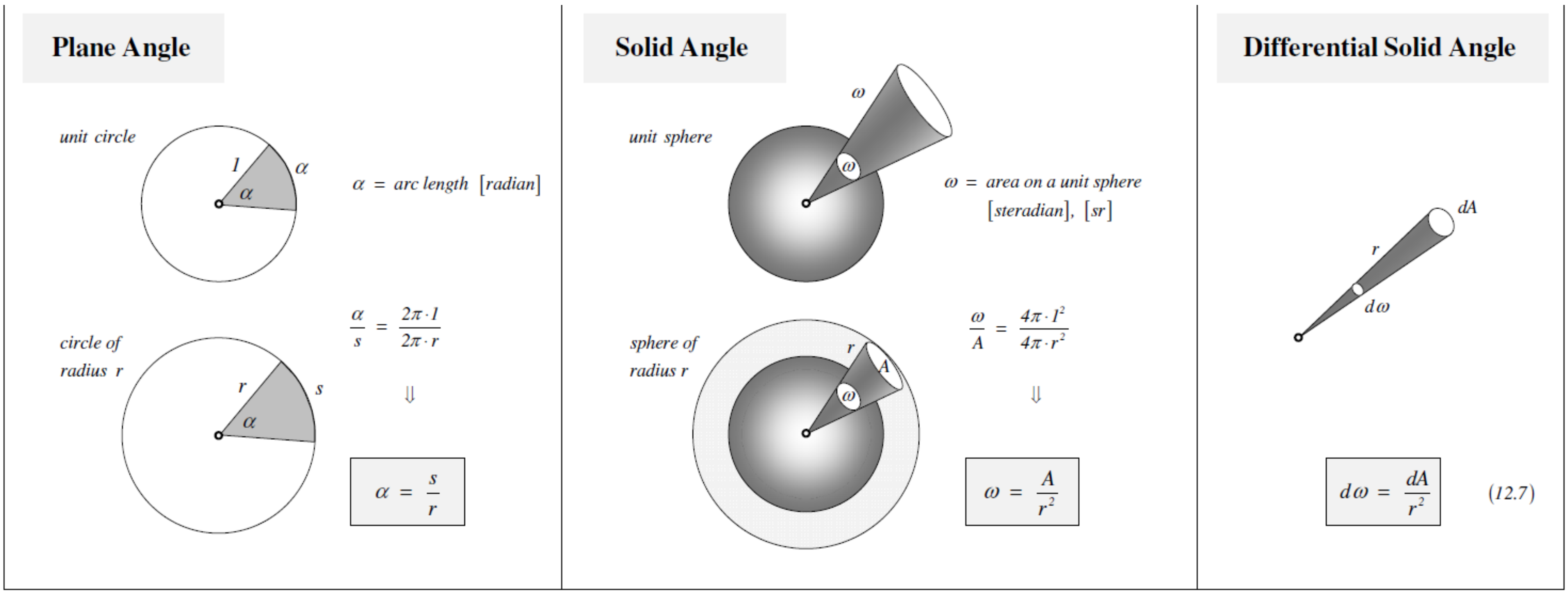
1. On slide 14, does the large curve that r(phi) points to represent the trajectory of the scattering particle in the center of mass frame? I'm having difficulty understanding what is being illustrated on that slide.

Your question – What is meant by solid angle? . Can you clarify the solid angle from lab vs. CM frames.

This is geometric construct that is useful especially to scattering theory.   The same construct applies to lab or CM frames.

From the webpage --

https://www.et.byu.edu/~vps/ME340/TABLES/12.0.pdf



Your question -- For defined interparticle potential the trajectory of a scattering particle is defined. So what physical meanings is the differential cross section?

Comment – The notion of cross section is motivated largely by the desire to compare experiment and theory. The theory gives us a trajectory and the experiment gives us data from the detector which we can standardize.

$$\text{Differential cross section}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector}$$

$$\text{at angle } \theta$$

Figure from F&W:

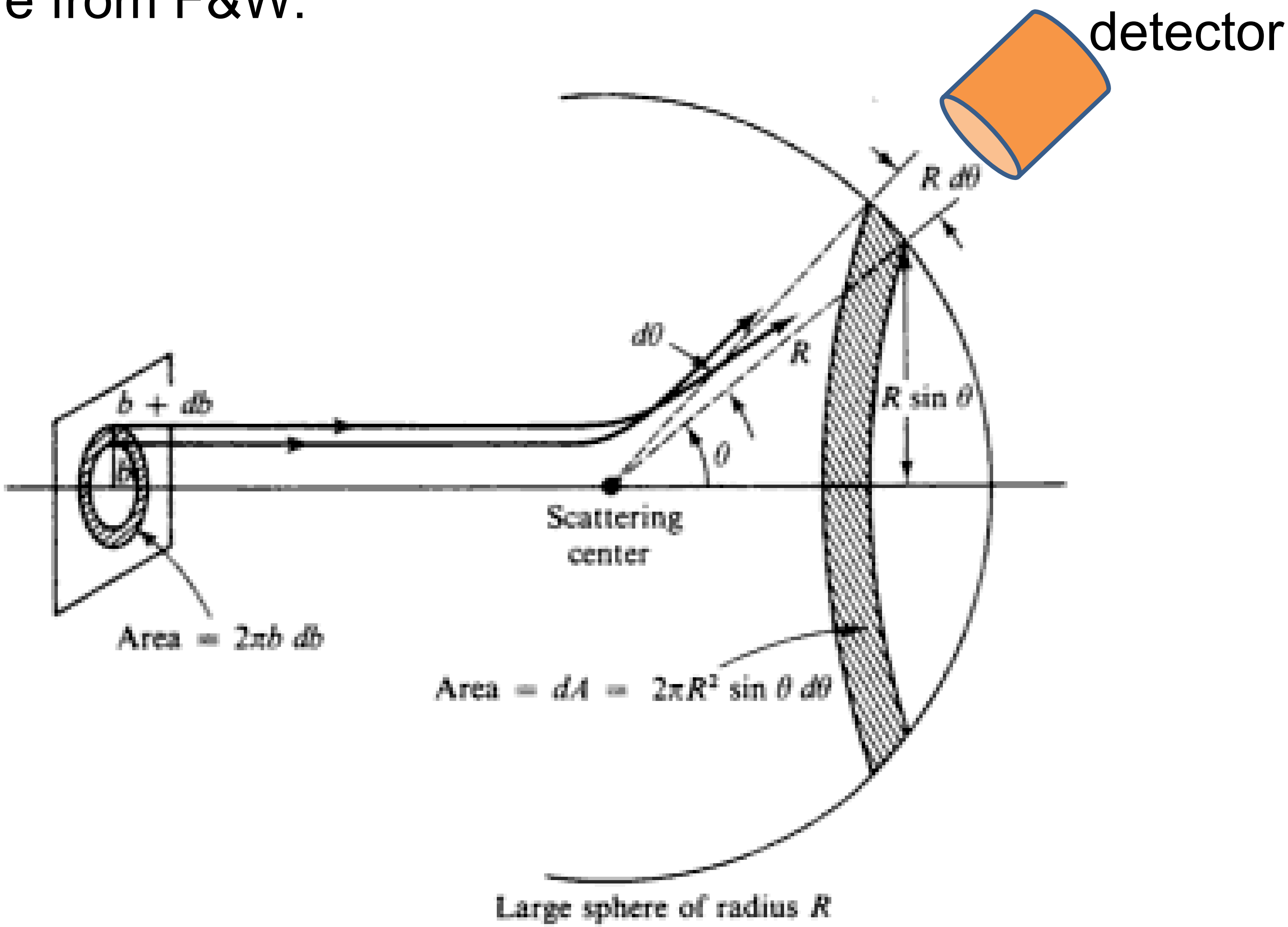
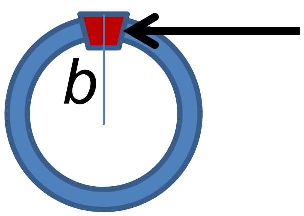


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

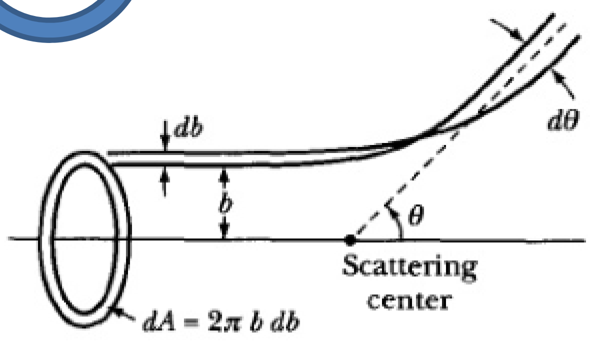
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$



$$d\sigma = d\phi b db$$

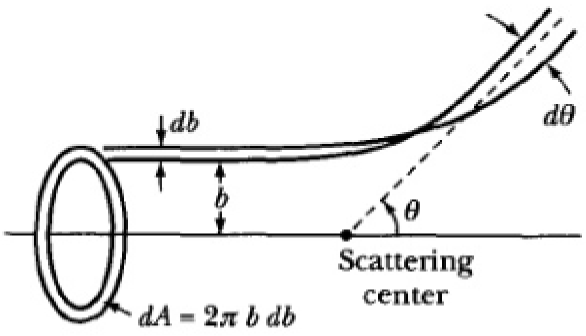
$$d\Omega = d\phi \sin\theta d\theta$$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: Notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the classical mechanics can we calculate it using geometric considerations



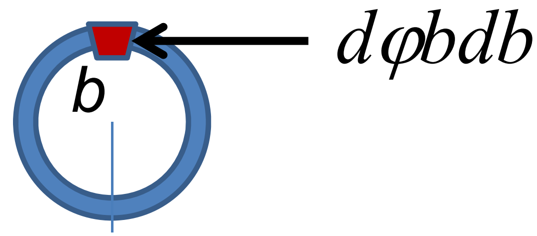


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Note in the above slides θ is the scattering angle in the lab frame and ϕ is the azimuthal angle measured in the plane perpendicular to the slide.   Unfortunately the textbook and the notes have some deviations in this notation

Transformation between center-of-mass and laboratory reference frames: (assuming that energy is conserved)

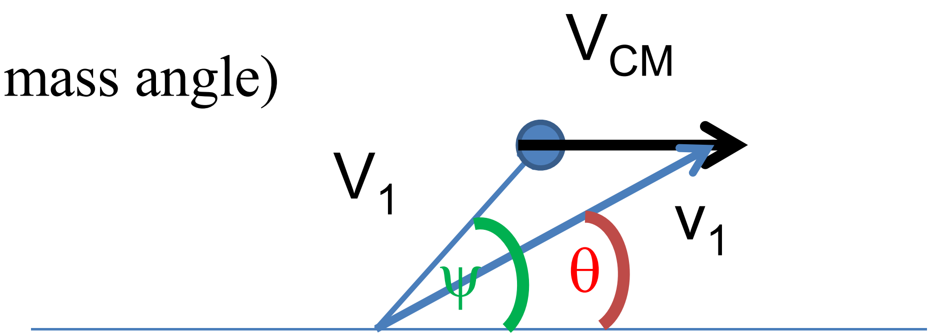
θ (lab angle) vs ψ (center of mass angle)

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$



Also: $$\cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2 \cos\psi + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right|=\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

For elastic scattering:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $$V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$$

Hard sphere: $$V(r) = \begin{cases} \infty & r \leq D \\ 0 & r > D \end{cases}$$ X

Coulomb or gravitational: $$V(r) = \frac{K}{r}$$ ←

Lennard-Jones: $$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

Total energy of system:

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V\left(r\right)$$

constants

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision.

Your question -- In E = E_CM + E _rel, what does rel mean? Relative?

Comment -- Yes

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$E = \quad E_{CM} \quad + \quad E_{relative}$$

Recall that $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Short hand notation:

$$\dot{r} \equiv \frac{dr}{dt}$$

Total energy of system:

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V\left(r\right)$$

$$E = \qquad E_{CM} \qquad + \qquad E_{rel}$$

Focus of analysis

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + V\left(r\right)}$$

$$= \frac{1}{2}\mu\dot{r}^2 + V_{eff}^{\ell}\left(r\right)$$

Comment on HW #2

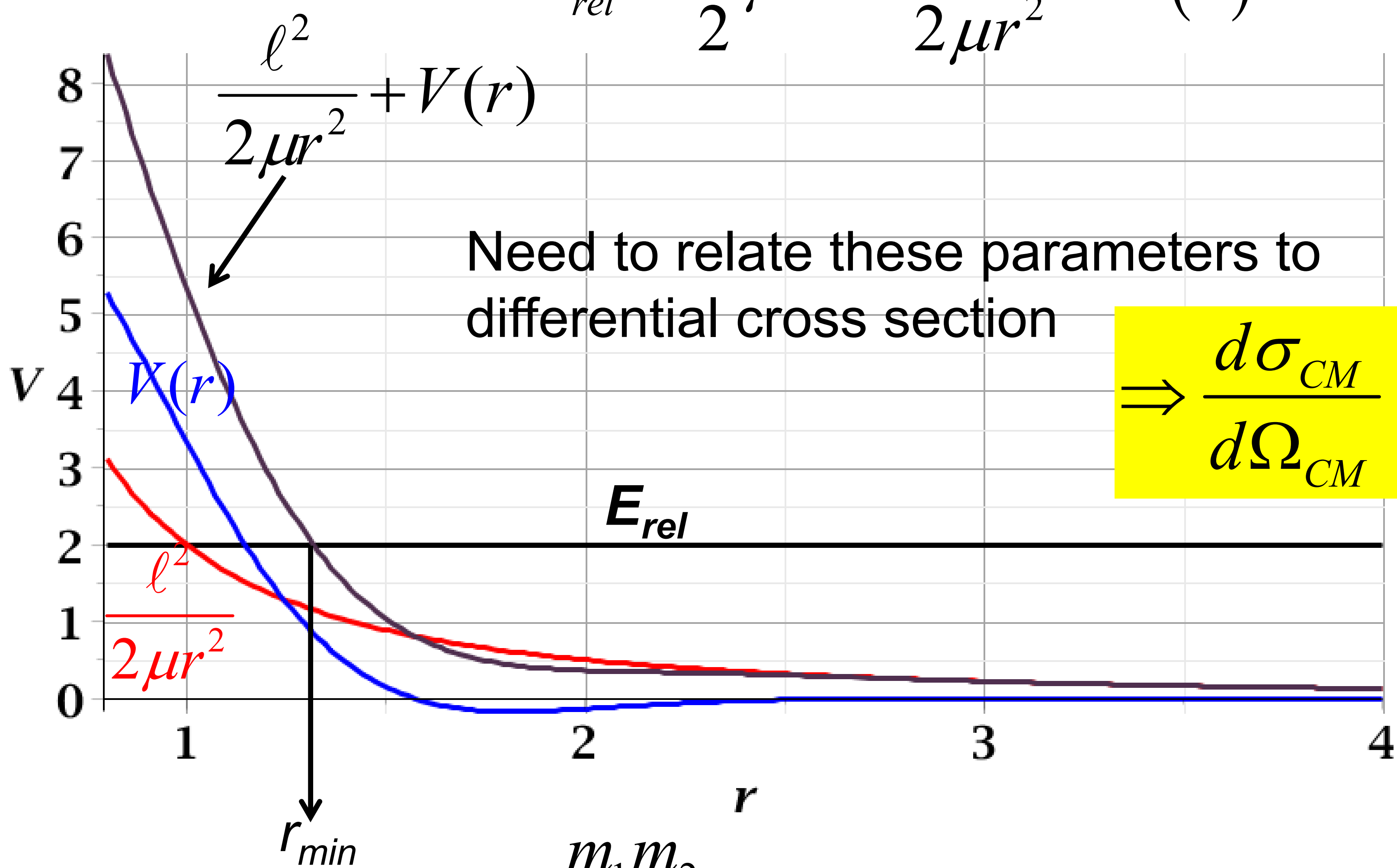
$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + V(r)}$$
$$= \frac{1}{2}\mu\dot{r}^2 + V_{eff}^{\ell}(r)$$

Note that we will show that

$$\frac{\ell^2}{2\mu r^2} = \frac{b^2 E_{rel}}{r^2}$$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section $\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

Your question -- So if you want your scattered particles to appear at a certain angle (theta), then you should play with the input parameter b until your particles are being picked up by your fixed detector? I guess the E*rel* could also be played with to vary K in the equation.

Comment – This is a very good question concerning what are the variables we have control over and which are fixed by the physics. In general whatever variables are fixed in the lab frame translates into particular variables in the center of mass frame. From that viewpoint, if you prepare a beam of particles with a given energy, that will fix E_{rel}. Typically the interaction parameters are fixed by the physics. The impact parameter *b* does vary throughout the beam profile and depending on the particle trajectories will result in various detector signals as a function of scattering angle.

Figure from F&W:

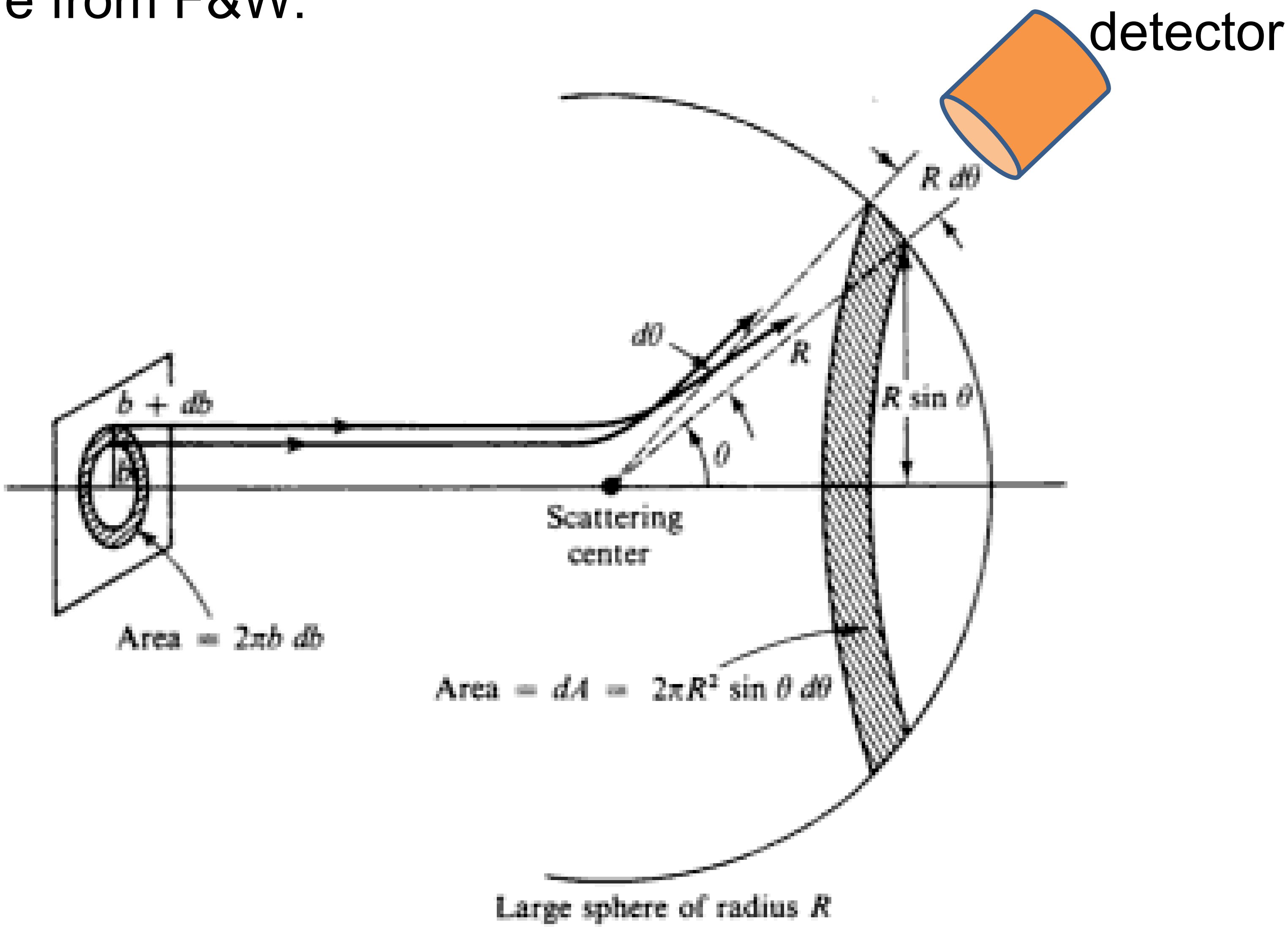
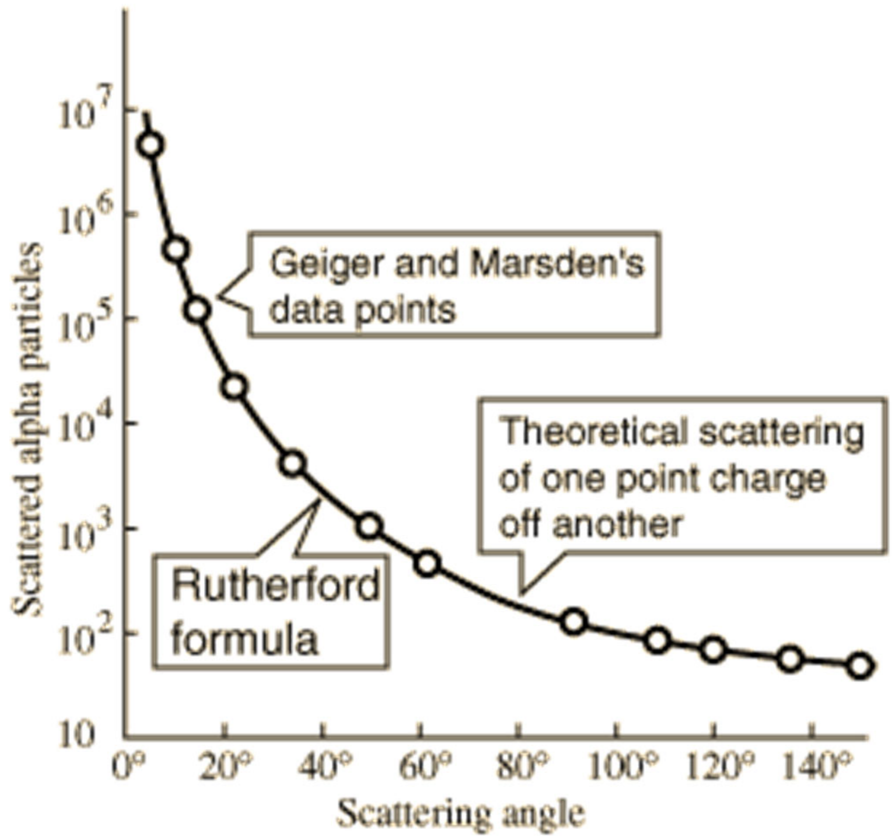


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



Example of data from Rutherford experiment

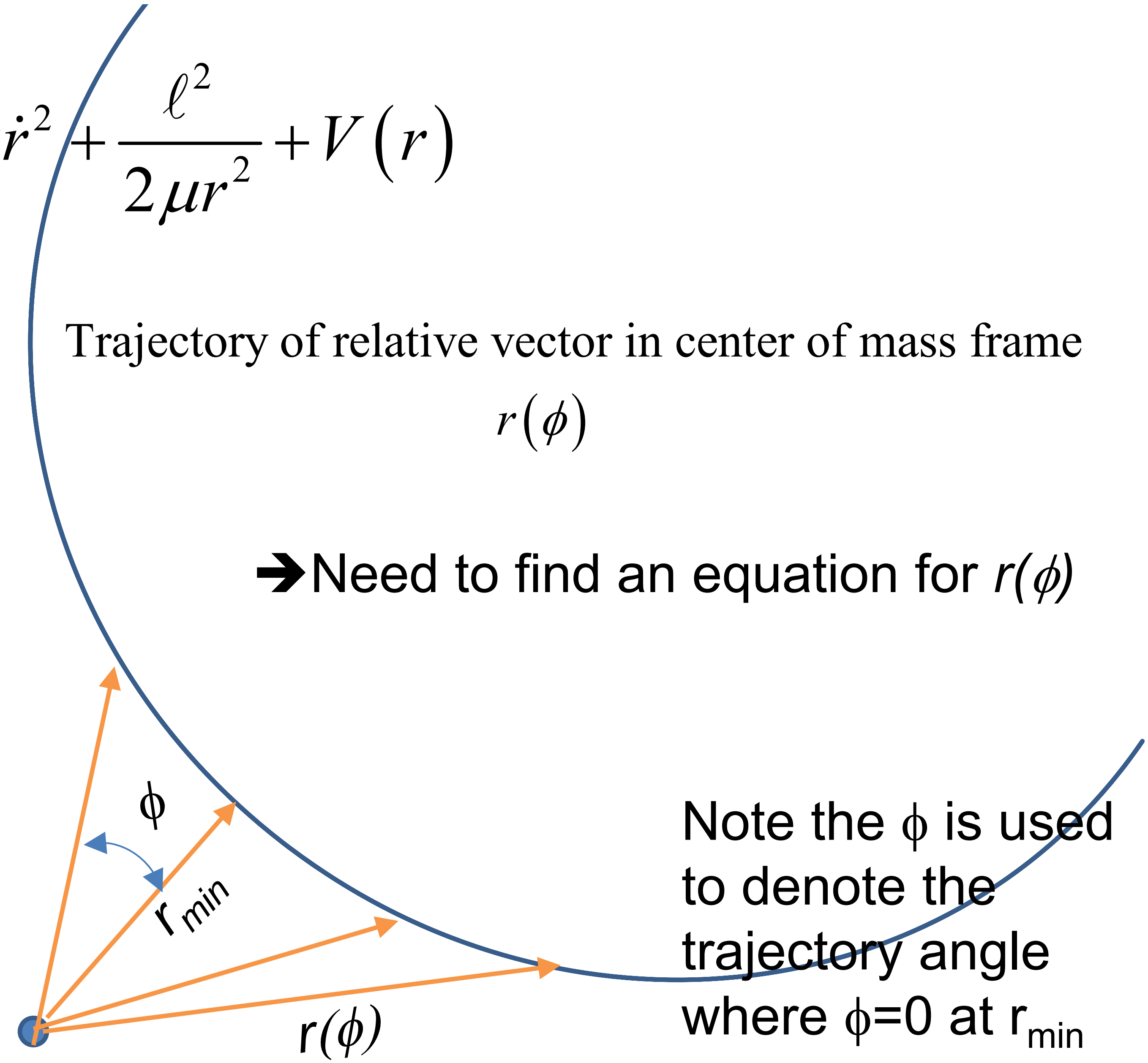
From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Trajectory of relative vector in center of mass frame

$$r(\phi)$$

➔Need to find an equation for *r(ϕ)*



Note the ϕ is used to denote the trajectory angle where ϕ=0 at r_{min}

center of scattering.

Your question -- On slide 14, does the large curve that r(phi) points to represent the trajectory of the scattering particle in the center of mass frame? I'm having difficulty understanding what is being illustrated on that slide.

Comment – My apologies for the figure and the notation. The intention was to relate to figures in your textbook such as Fig. 5.3 where now ϕ is an angle associated with the trajectory $r(\phi)$. The point is, while we can analyze the details of the full trajectory from the equations, for scattering theory we only need to know what happens before and after the particle gets close to the target.

Questions:

1. How can we find $r(\varphi)$?
2. If we find $r(\varphi)$, how can we relate φ to ψ?
3. How can we find $b(\psi)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$

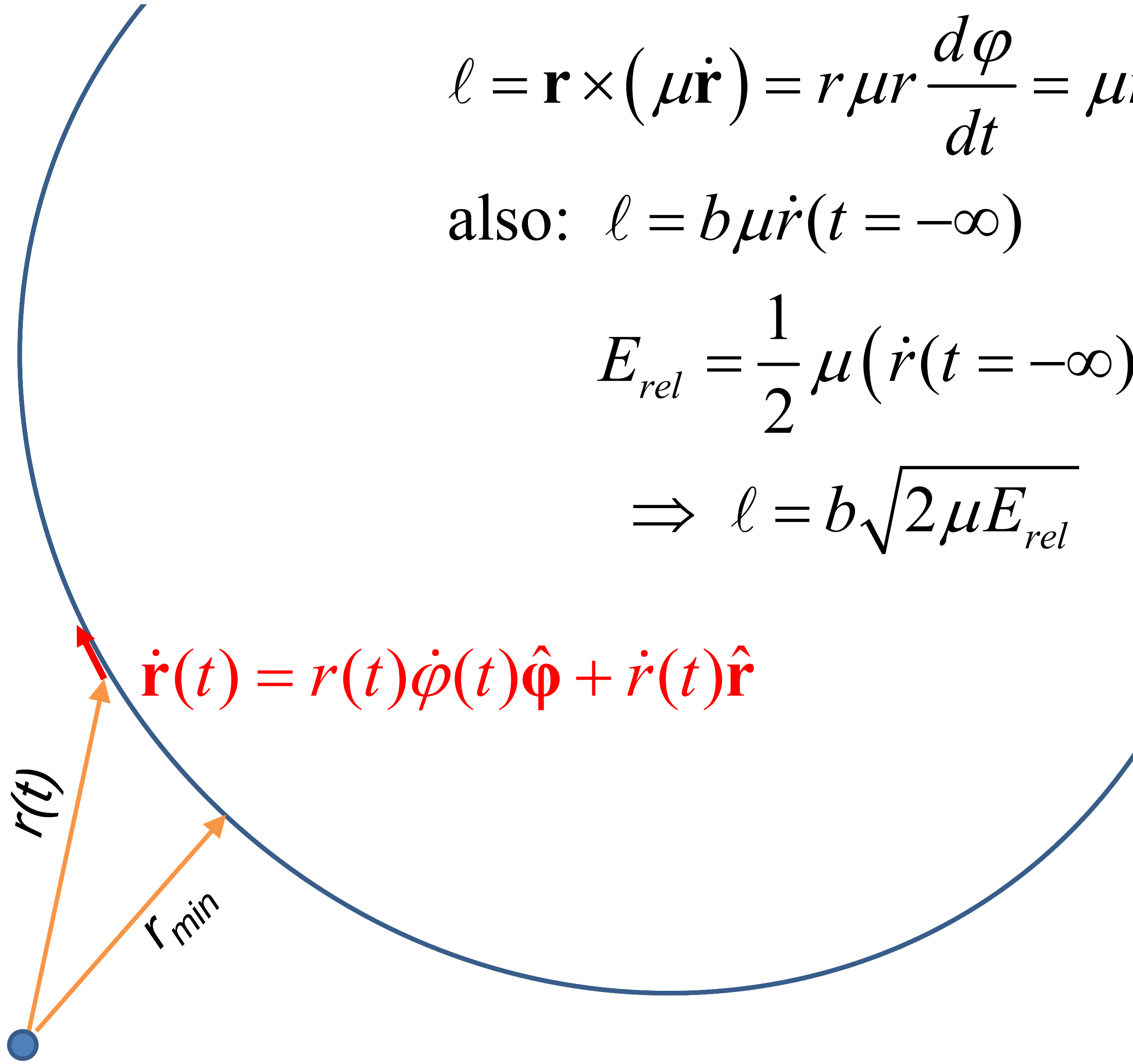
$$\ell = \mathbf{r} \times \left(\mu \dot{\mathbf{r}} \right) = r \mu r \frac{d\varphi}{dt} = \mu r^2 \frac{d\varphi}{dt}$$

also: $\ell = b \mu \dot{r}(t = -\infty)$

$$E_{rel} = \frac{1}{2} \mu \left(\dot{r}(t = -\infty) \right)^2$$

$$\Rightarrow \quad \ell = b \sqrt{2 \mu E_{rel}}$$

$$\dot{\mathbf{r}}(t) = r(t) \dot{\varphi}(t) \hat{\boldsymbol{\varphi}} + \dot{r}(t) \hat{\mathbf{r}}$$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\varphi)$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2 \left(\frac{d\varphi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\varphi) \Leftrightarrow \varphi(r)$:

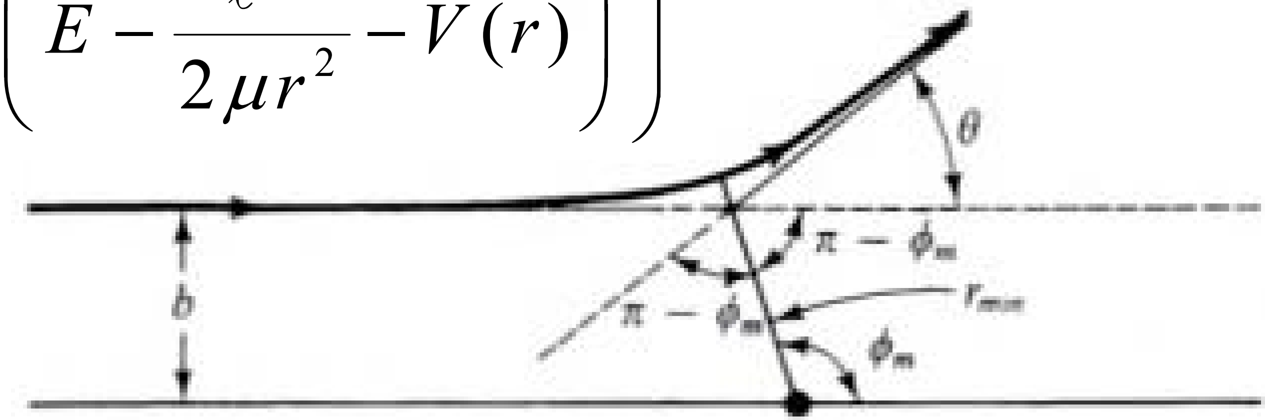
From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

v_∞ $\longrightarrow$



Special values at large separation ($r \rightarrow \infty$):

$$\ell = \mu\left|\mathbf{r}\times\mathbf{v}\right|_{r\rightarrow\infty} = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}\,b$$

When the dust clears:

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \dfrac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{b / r^2}{\sqrt{1 - \dfrac{b^2}{r^2} - \dfrac{V(r)}{E}}}\right)$$

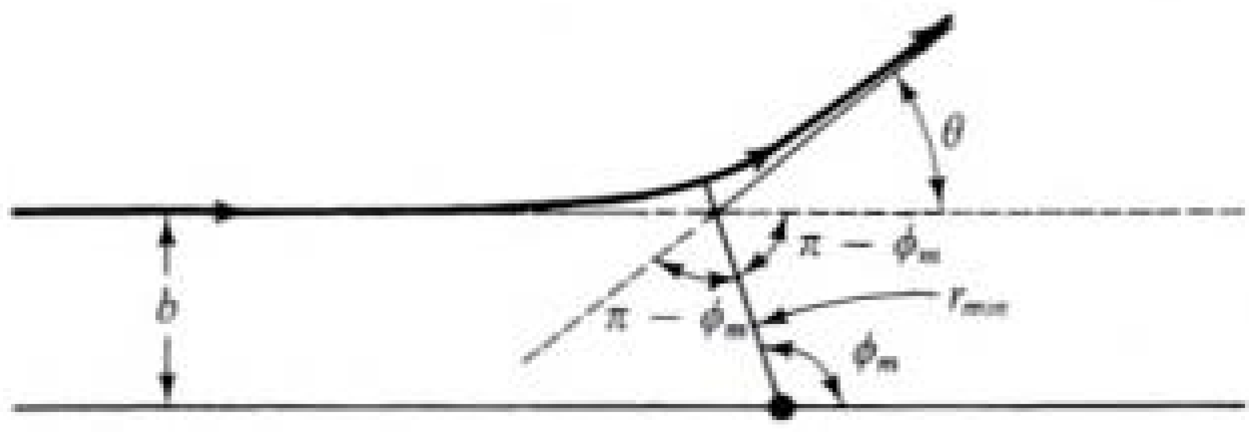
$$\Rightarrow \varphi_{\max}(b, E) = \varphi\left(r \to \infty\right) - \varphi\left(r = r_{\min}\right)$$

$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where :

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Relationship between ϕ_{max} and θ:



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

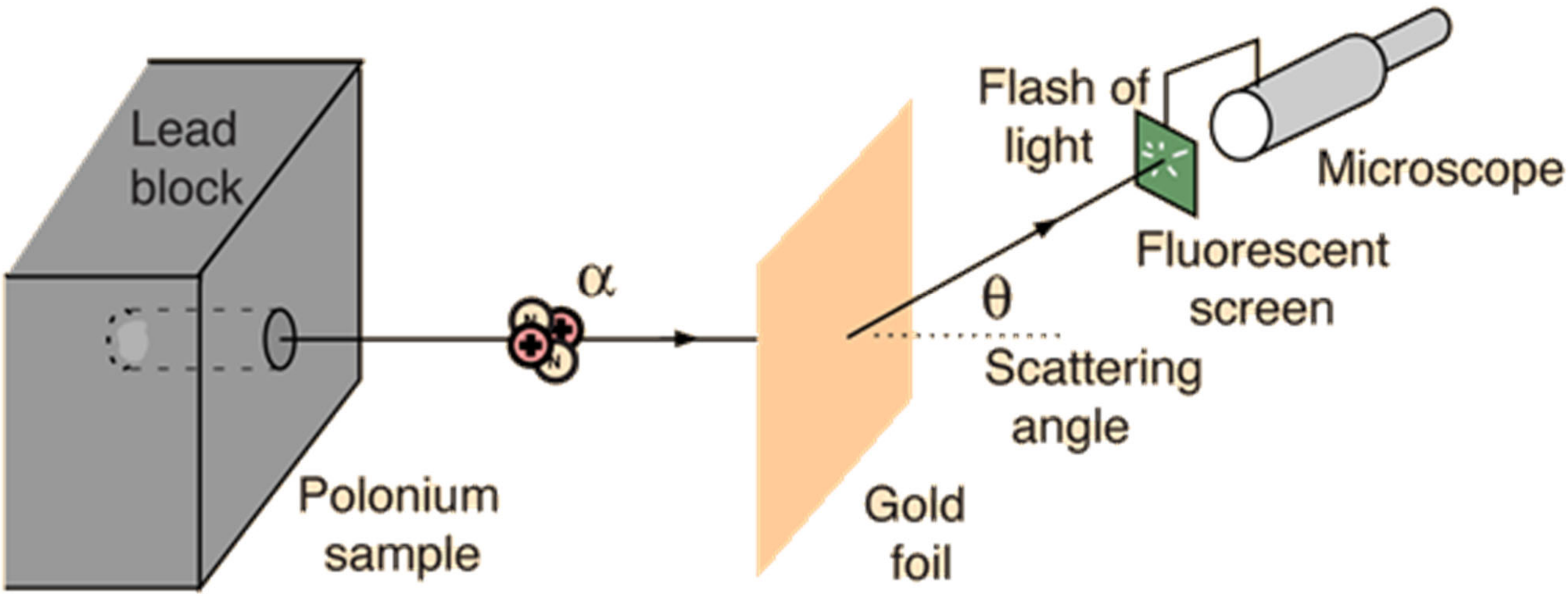
Using the diagram from your text, θ represents the scattering angle in the center of mass frame.

$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

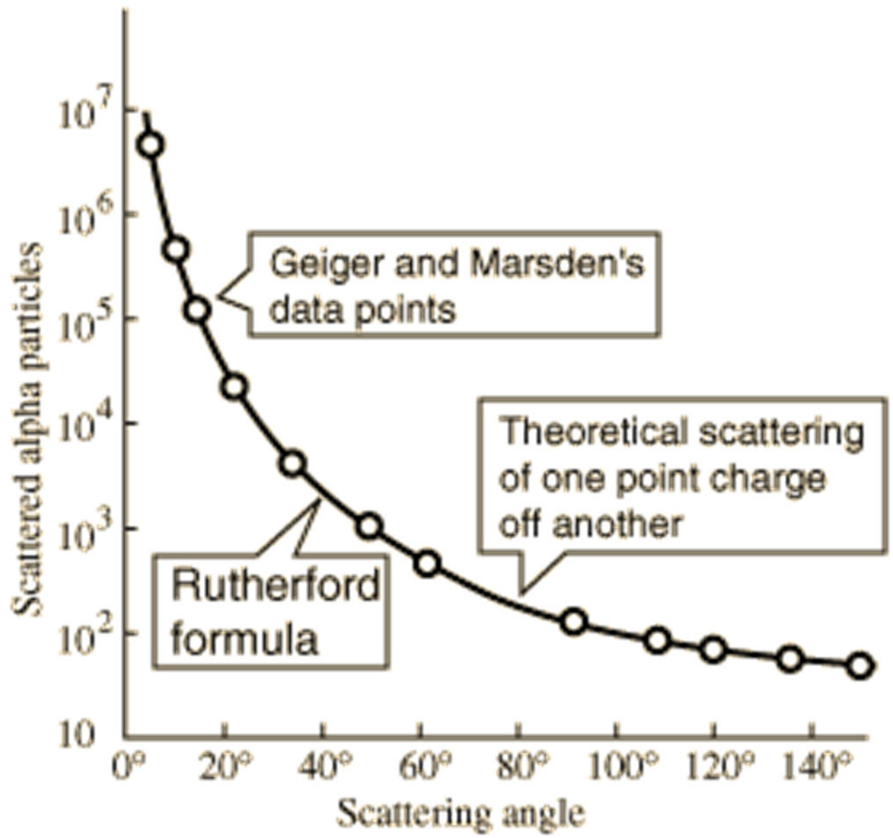
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\theta = -\pi + 2b\int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where $r_{\min}$ is found from

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

In preparation for Friday's lecture, please continue to review these slides and pose your questions (before 7 AM on Friday). Based on your questions and where we have arrived in today's lecture, hopefully we can help you have some understanding of scattering theory.