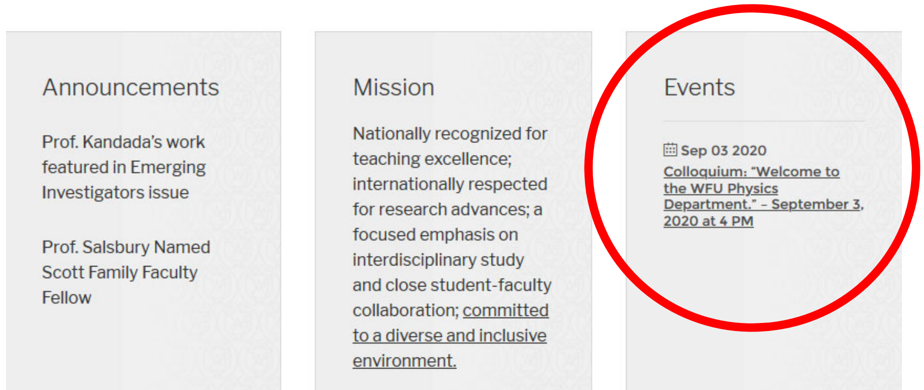
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Plan for Lecture 4 -- Chapter 1 F&W

1. Summary of what we have learned so far
2. Analytical evaluation of the differential scattering cross section in general and for Rutherford scattering (in center of mass frame)

In this lecture, we will focus on the analysis in the center of mass frame which contains the basic physics of the interparticle interaction. The main example will be that of the famous Rutherford scattering experiment.

Physics colloquium Thursday at 4 PM –
https://www.physics.wfu.edu/



This week the colloquium series will be starting. The information is posted on our website

Colloquium: “Welcome to the WFU Physics Department.” – September 3, 2020 at 4 PM

PROGRAM

The purpose of this first seminar is to help new, returning, and prospective students (including both undergraduate and graduate students), faculty, and staff to become acquainted with each other and with the Physics Department. We will meet online at 4:00 PM for presentations by some undergraduate students highlighting their summer research experiences, followed by general welcoming statements and departmental announcements. We are especially pleased to welcome quite a few new additions to the physics department.

Date
Sep 03 2020

Time
4:00 pm - 5:00 pm

CATEGORY
› Colloquium

Share this event

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Here we see the details for the first colloquium which will introduce you to the department and various important announcements.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory		
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation		

Here is a snapshot of the lecture and homework schedule. From your discussion, I slowed the pace a bit, scheduling one more session on scattering so that we can summarize and digest. There is no new homework, so that by Friday, we will hopefully be all caught up.

Scattering theory:

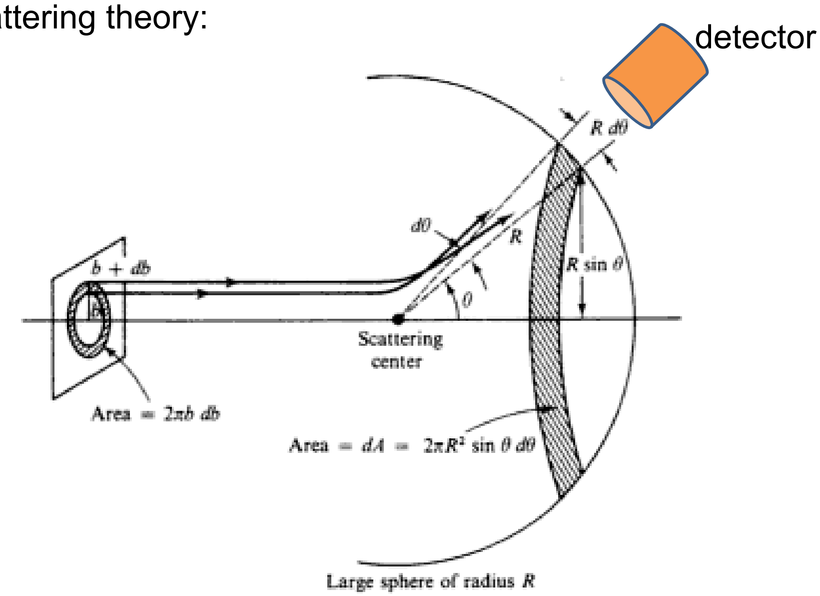

Figure 5.5 The scattering problem and relation of cross section to impact parameter.

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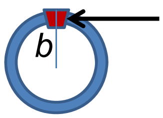
This and the next several slides summarize the experimental scattering geometry.

Differential cross section

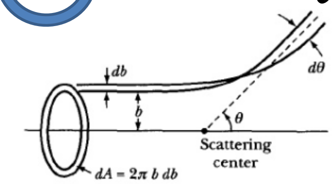
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$



$$d\sigma = d\phi b db$$

$$d\Omega = d\phi \sin\theta d\theta$$

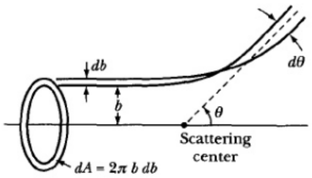


$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

More review.

Note: Notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the classical mechanics can we calculate it using geometric considerations



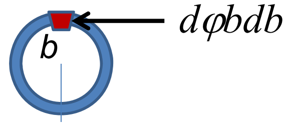


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

More review.

Transformation between center-of-mass and laboratory reference frames: (assuming that energy is conserved)

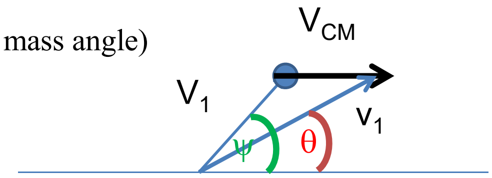
θ (lab angle) vs ψ (center of mass angle)

$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$

$v_1 \sin\theta = V_1 \sin\psi$

$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$



Also: $$\cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\psi + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right|=\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

For elastic scattering:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

$$\text{where:}\quad \tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

Review of equations relating cross section results in center of mass frame to that in the laboratory experiment.

Focusing on the center of mass frame of reference:

Typical two-particle interactions –

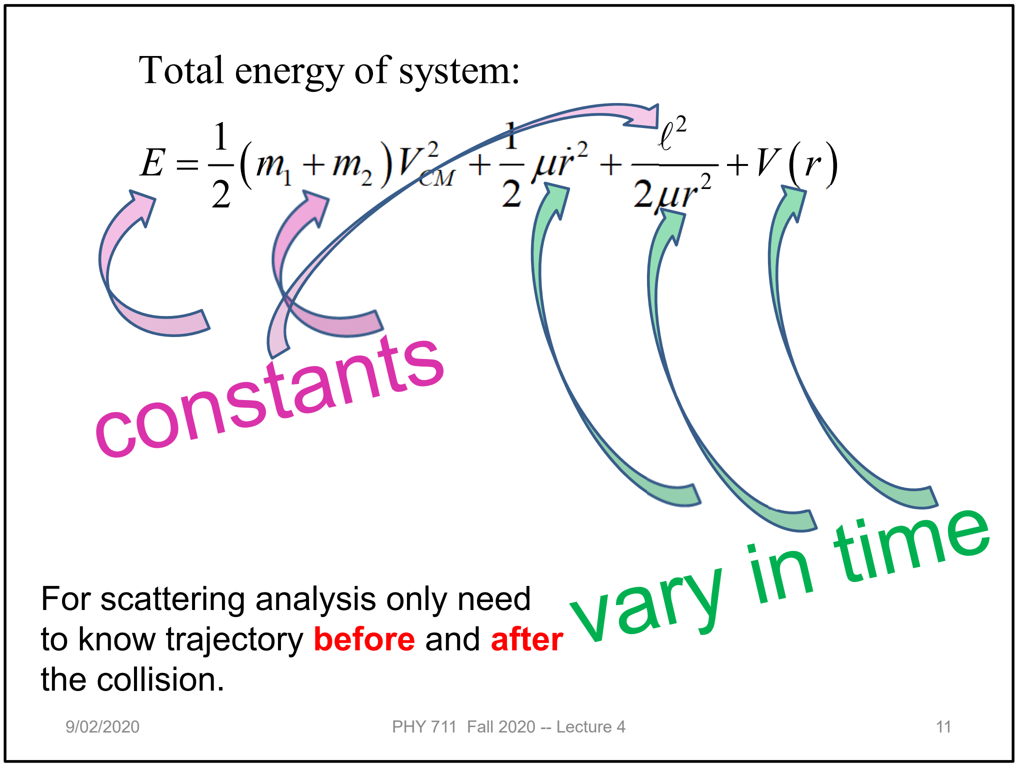
Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \leq D \\ 0 & r > D \end{cases}$ X

Coulomb or gravitational: $V(r) = \frac{K}{r}$ ←

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

Now we will focus our attention on the analysis of the event in the center of mass frame of reference and particularly consider the Coulomb interaction (which also has the same functional form as the gravitional interaction.



Recall that the system of interest can be expressed in terms of constants and the particle separation r(t).

Short hand notation:

$$\dot{r} \equiv \frac{dr}{dt}$$

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$E = \quad E_{CM} \quad + \quad E_{rel}$$

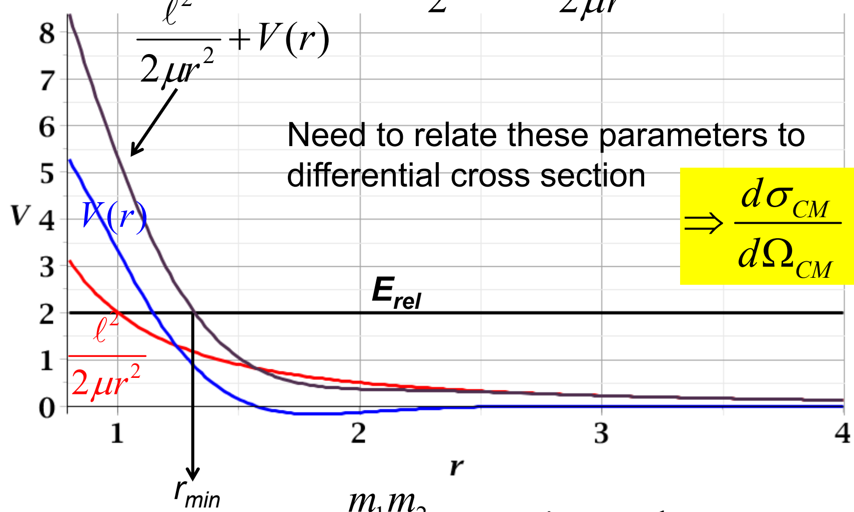
Focus of analysis

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + V(r)}$$

$$= \frac{1}{2}\mu\dot{r}^2 + V_{eff}^{\ell}(r)$$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section $\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

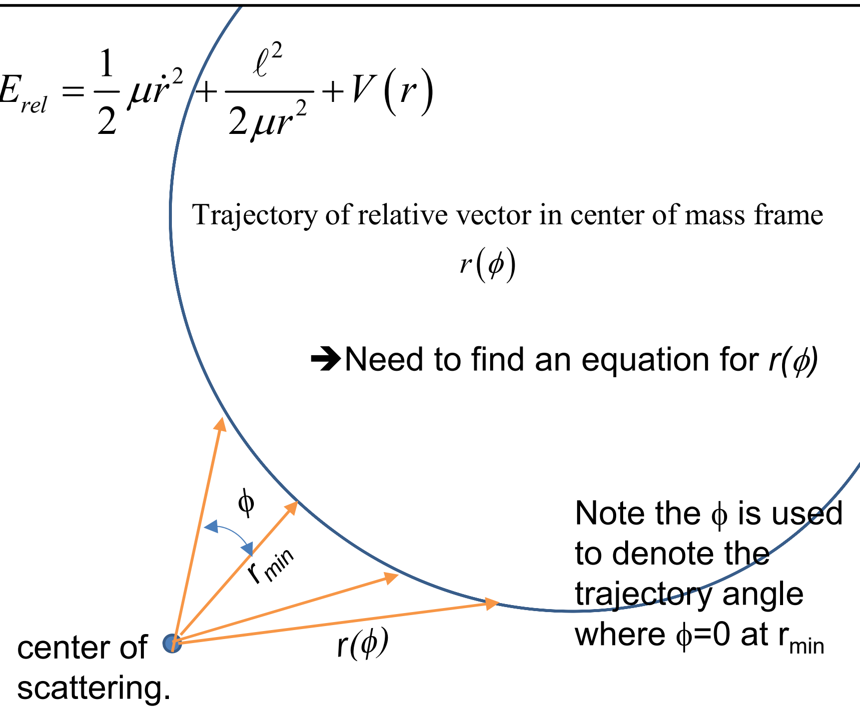
ℓ=angular momentum

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Trajectory of relative vector in center of mass frame

$r(\phi)$

➔Need to find an equation for $r(\phi)$

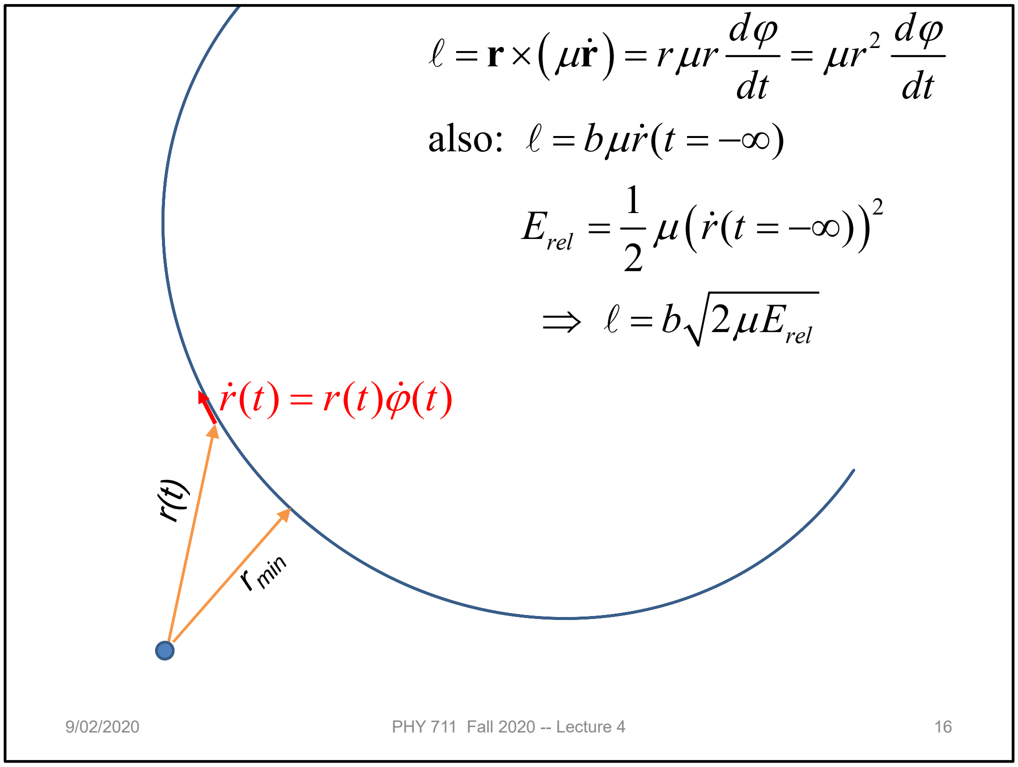


Note the ϕ is used to denote the trajectory angle where ϕ=0 at r_{min}

Questions:

1. How can we find $r(\varphi)$?
2. If we find $r(\varphi)$, how can we relate φ to ψ?
3. How can we find $b(\psi)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$



Relating the constants of the problems to values determined when the particles are very far apart.

Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\varphi)$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\varphi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Up to now we are expressing the trajectory as a function of time t. For this analysis however, we would like to find the trajectory as a function of the angle \phi. The relationship of the constant angular momentum to the \phi and time variables allows us to find r(\phi).

Solving for $r(\varphi) \Leftrightarrow \varphi(r)$:

From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

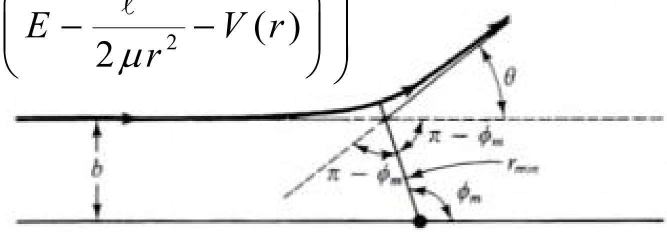
$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

Lots of algebra ….

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$v_\infty \longrightarrow$



Special values at large separation ($r \to \infty$):

$$\ell = \mu \left|\mathbf{r} \times \mathbf{v}\right|_{r\to\infty} = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}\, b$$

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This shows the relationship of the analysis of r(\phi) to the scattering event. The insert is a diagram from the textbook.

When the dust clears:

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\Rightarrow \varphi_{\max}(b,E) = \varphi\left(r \to \infty\right) - \varphi\left(r = r_{\min}\right)$$

The scattering process involves the aspects of the trajectory at large separations.

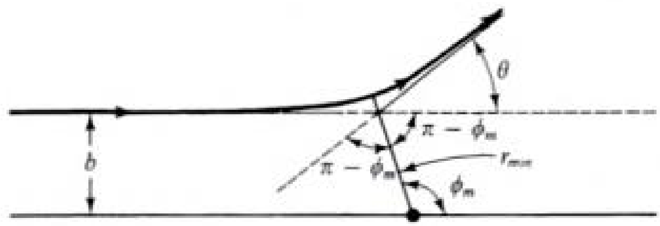
$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where :

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

When r=rmin, \phi=0. \phi increases to a maximum value.

Relationship between ϕ_{max} and θ:



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

Using the diagram from your text, θ represents the scattering angle in the center of mass frame.

Problem with consistent notation. Here we are using the textbook's definition of theta which is OK if the target mass is infinite.

$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1 / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

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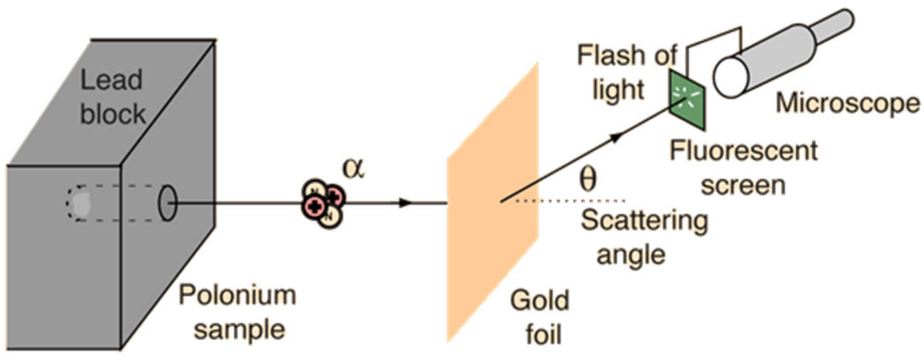
The last equation is a convenient change of variables.

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Here we use a diagram from the web about the Rutherford geometry.

Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2\mathrm{b}} + \sqrt{\left(\frac{\kappa}{2\mathrm{b}}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

Summary of all of the equations and then specializing to the Coulomb interaction form.

Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

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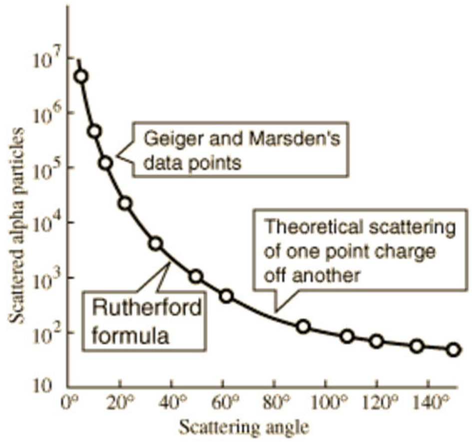
In this case, the integrals can all be evaluated analytically and simplify to this famous result.

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Reconstructed data taken from the web.

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

Evaluation of constant for this case.

Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\theta = -\pi + 2b\int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

where $r_{\min}$ is found from

$$1-\frac{b^2}{r_{\min}^2}-\frac{V(r_{\min})}{E} = 0$$

Recap of the equations. This equation allows us to fine b(theta).