PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Discussion notes for Lecture 5

Wrap up of scattering analysis – Chap 1 F&W

1. **Summary of equations in the center of mass frame.**
2. **In center of mass frame, analytical evaluation of the differential scattering cross section in general and for Rutherford scattering.**
3. **Summary of results including transformation to lab frame.**

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday   (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi – 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation		

For Monday, please read the annotated lecture notes for Lecture 6 (available <7 AM on 9/6/2020). There will be no homework for Monday's lecture, leaving more time for HW #4.

PHY 711 -- Assignment #4

Finish reading Chapter 1 in **Fetter & Walecka**.

1. Work Problem #1.16 at the end of Chapter 1 in **Fetter and Walecka**.

Your questions –
From Tim & Nick

1. Questions about HW #3

From Jeanette

1. In E = E_CM + E _rel, what does rel mean? Relative?
2. Slide 16 - E_rel only has one term, but in slide 14 it had 3 terms. What happened to the other 2 terms?

From Bamidele

Slide 13: What conditions made the *m*2 become ½ μ|*v*1−*v*2|2 *;*
μ=*m*1*m*2/(*m*1+*m*2)

Slide 14: Where does the term for the kinetic energy from angular momentum come from.

From Nick

1. The last line on slide 20 of lecture 4...can you clarify?

From Derek

1. On slide 14, does the large curve that r(phi) points to represent the trajectory of the scattering particle in the center of mass frame? I'm having difficulty understanding what is being illustrated on that slide.

Reviewing – scattering in the laboratory

Figure from F&W:

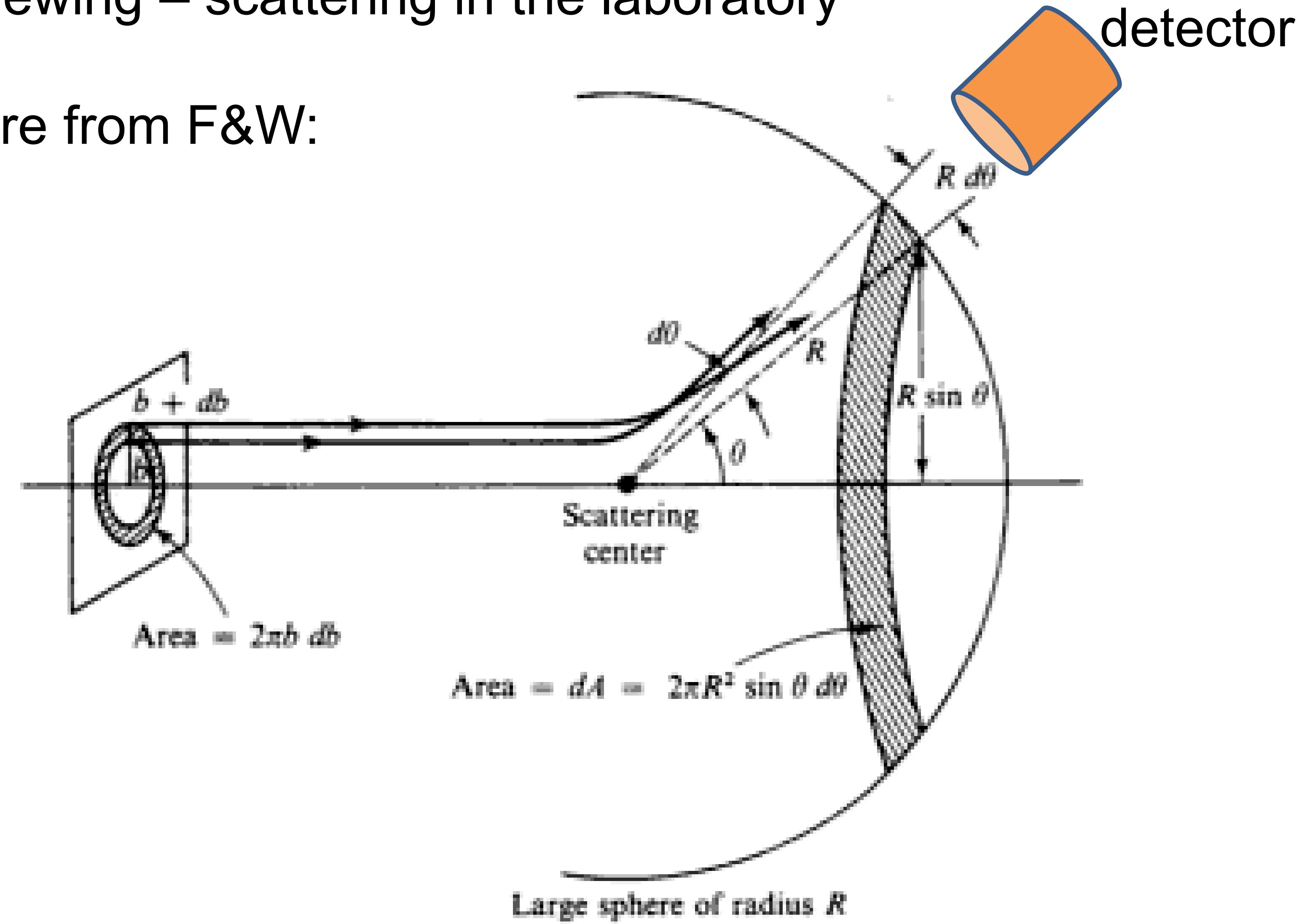
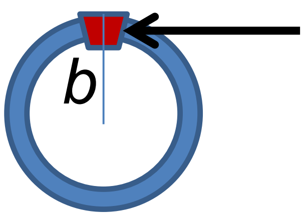


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

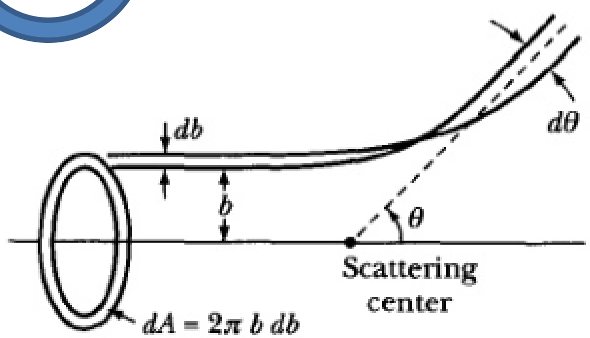
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$



$$d\sigma = d\varphi b db$$

$$d\Omega = d\varphi \sin\theta d\theta$$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constants

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision.

Some details --

Relationship between center of mass and laboratory frames of reference. At and time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \left(m_1 + m_2\right)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \left(m_1 + m_2\right)\dot{\mathbf{R}}_{CM} = \left(m_1 + m_2\right)\mathbf{V}_{CM}$$

Note that $\dot{\mathbf{R}}_{CM} \equiv \frac{d\mathbf{R}_{CM}}{dt} \equiv \mathbf{V}_{CM}$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

$$= \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

More details

Total energy of system:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E = \frac{1}{2}(m_1 + m_2) V_{CM}^2 + \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$E = \quad E_{CM} \quad + \quad E_{relative}$$

Recall that $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Since **r**(t) represents motion in a plane, we will analyze the system in that plane and use polar coordinates.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

$$x(t) = r(t)\cos(\phi(t))$$

$$y(t) = r(t)\sin(\phi(t))$$

Note that $\left|\dot{\mathbf{r}}(t)\right|^2 = \dot{x}^2(t) + \dot{y}^2(t)$

$$= \dot{r}^2(t) + r^2(t)\dot{\phi}^2(t)$$

Also note that the relative angular momentum of the system is a constant

$\ell = \mu r^2 \dot{\phi}$

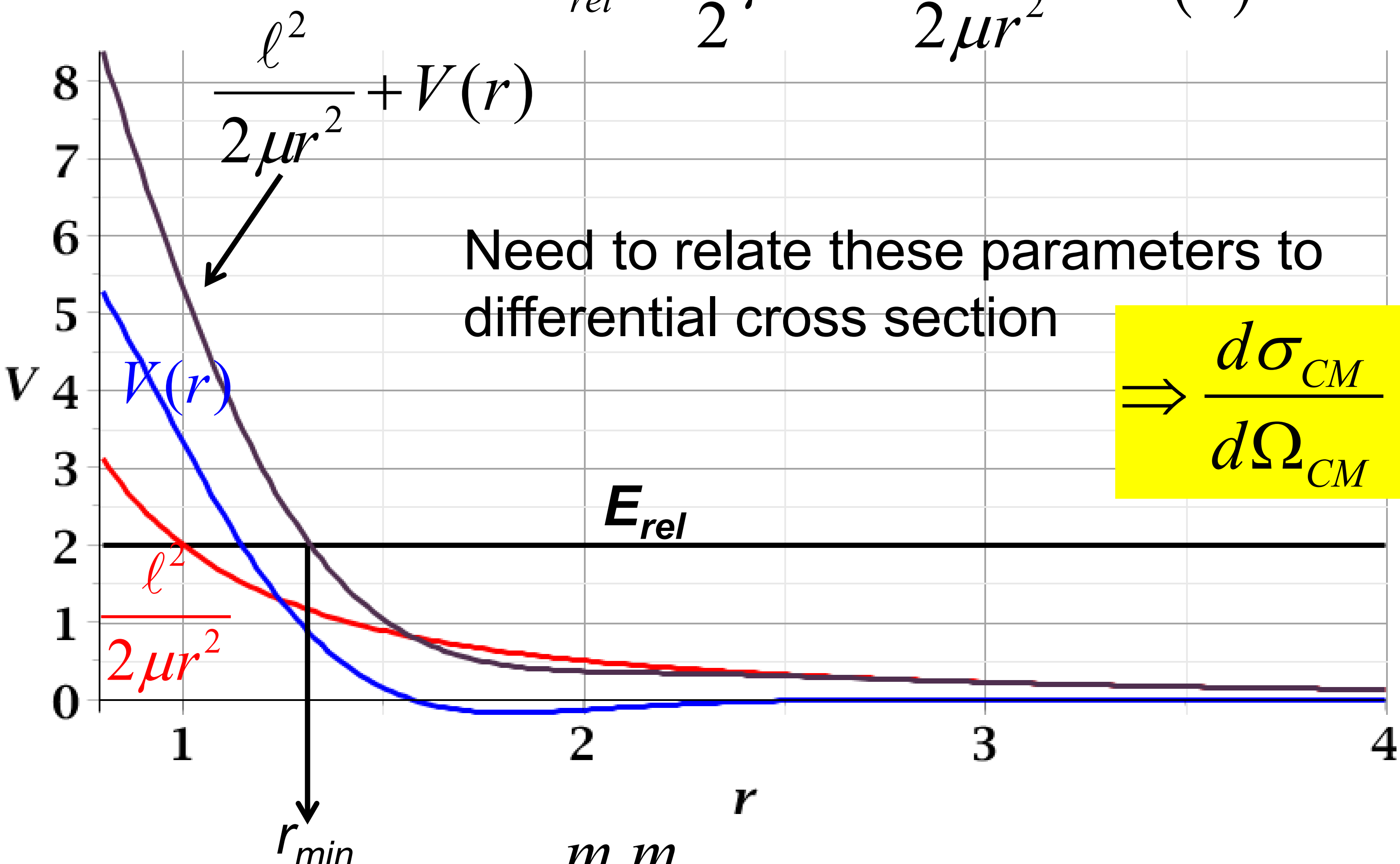
So that $\frac{1}{2}\mu \;|\dot{\mathbf{r}}(t)|^2 = \frac{1}{2}\mu\left(\dot{r}^2(t) + r^2(t)\dot{\phi}^2(t)\right)$

$$= \frac{1}{2}\mu\dot{r}^2(t) + \frac{\ell^2}{2\mu r^2}$$

$\Rightarrow$ $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



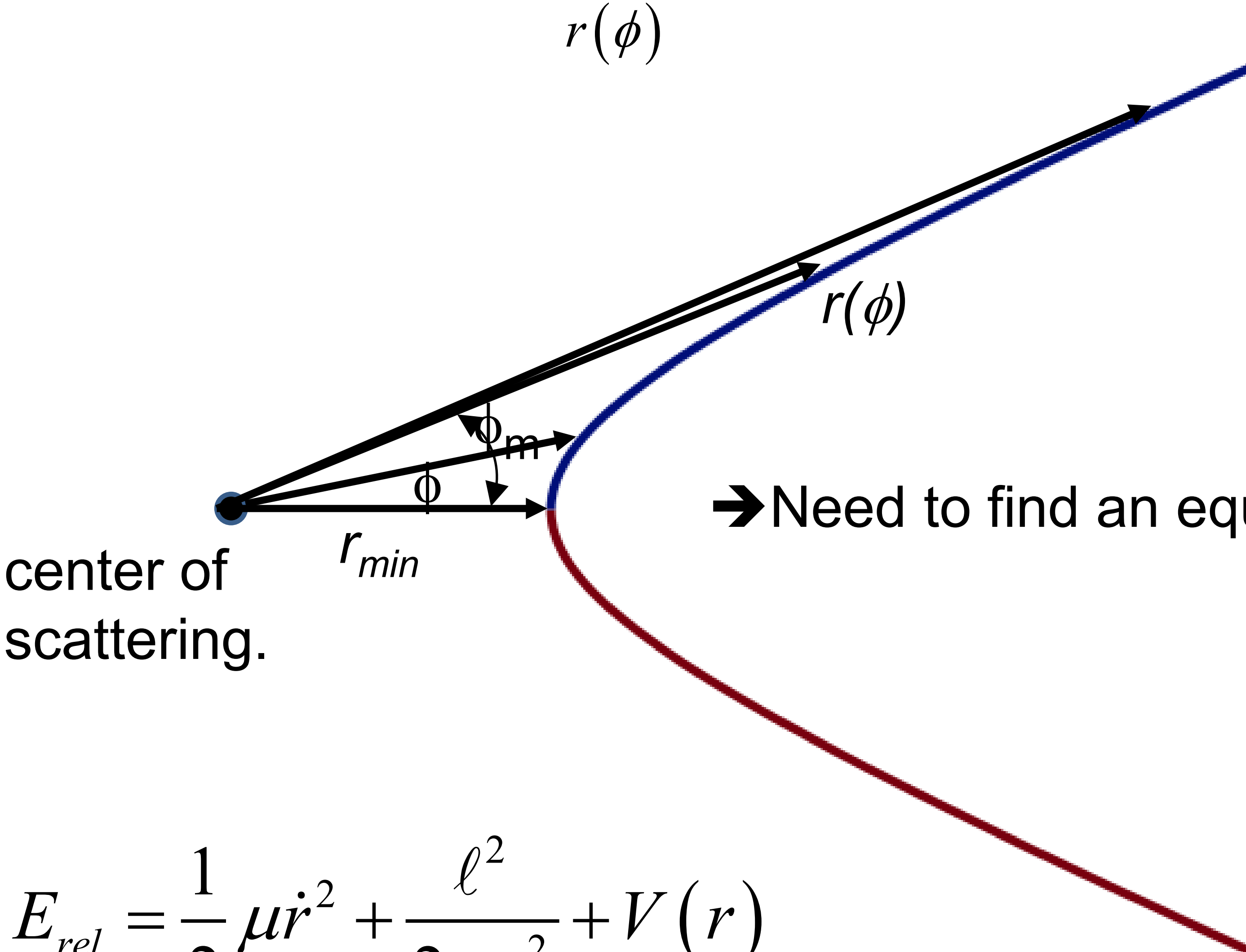
Need to relate these parameters to differential cross section

$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

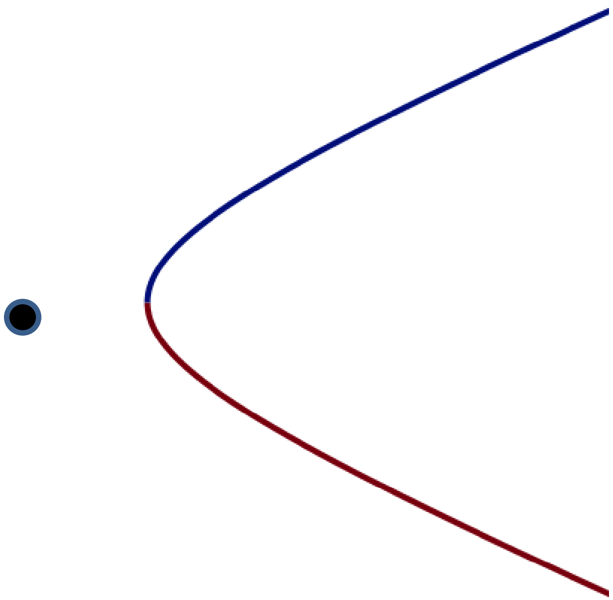
Trajectory of relative vector in center of mass frame



➔Need to find an equation for $r(\phi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

How is this related to scattering?



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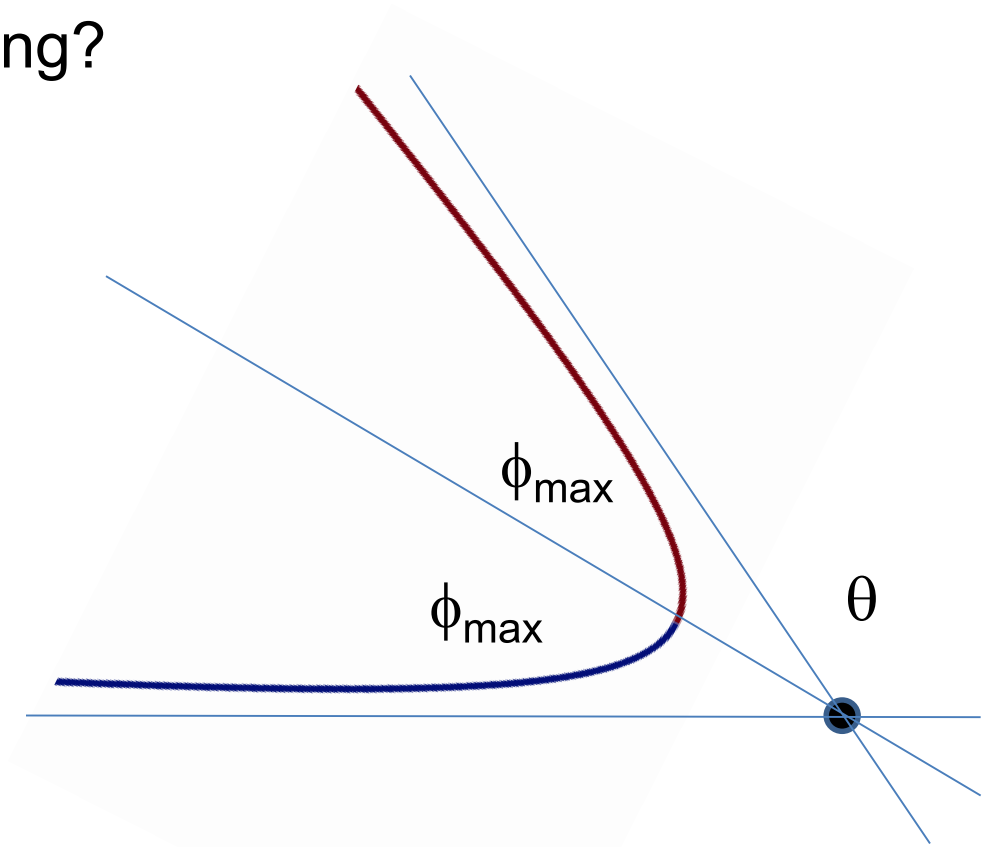


Note that here θ is used for the scattering angle

Questions:

1. How can we find $r(\varphi)$?
2. If we find $r(\varphi)$, how can we relate φ to ψ?
3. How can we find $b(\psi)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$

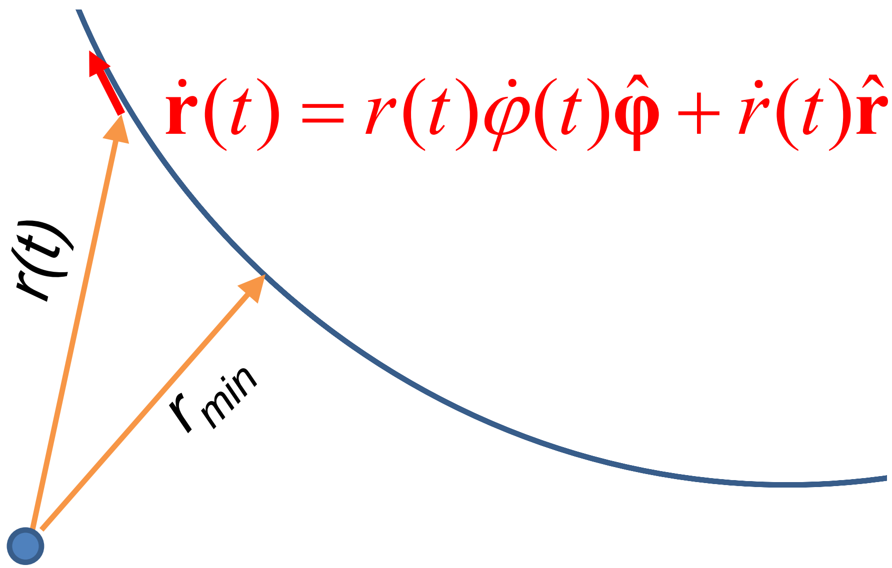
Evaluation of constants far from scattering center --

$$\ell = \mathbf{r} \times (\mu \dot{\mathbf{r}}) = r \mu r \frac{d\varphi}{dt} = \mu r^2 \frac{d\varphi}{dt}$$

$$\text{also: } \ell = b \mu \dot{r}(t = -\infty)$$

$$E_{rel} = \frac{1}{2} \mu \left(\dot{r}(t = -\infty) \right)^2$$

$$\Rightarrow \ell = b\sqrt{2\mu E_{rel}}$$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\varphi)$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2 \left(\frac{d\varphi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\varphi) \Leftrightarrow \varphi(r)$:

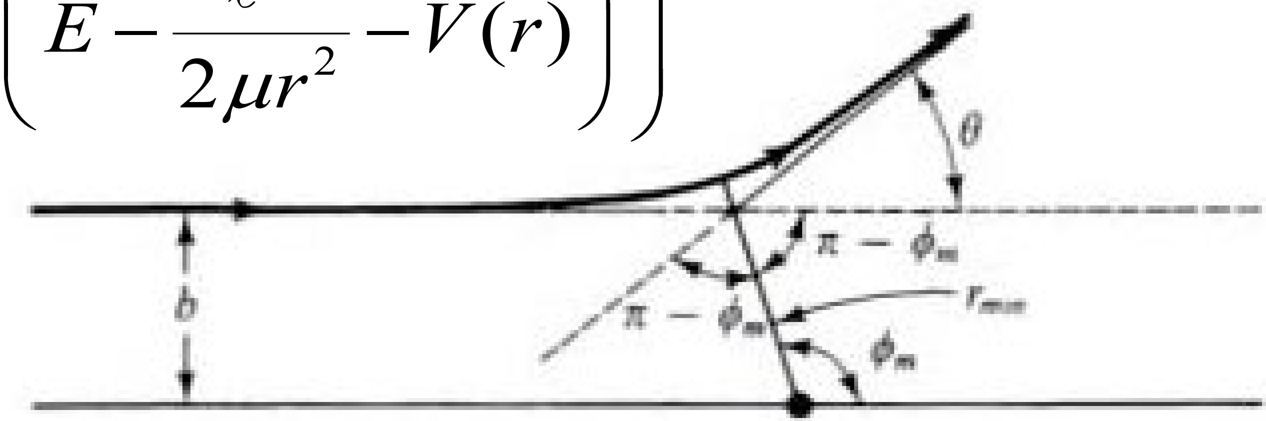
From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\varphi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

v_∞ →



Special values at large separation ($r \to \infty$):

$$\ell = \mu\left|\mathbf{r} \times \mathbf{v}\right|_{r\to\infty} = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}\, b$$

When the dust clears:

$$d\varphi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\varphi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

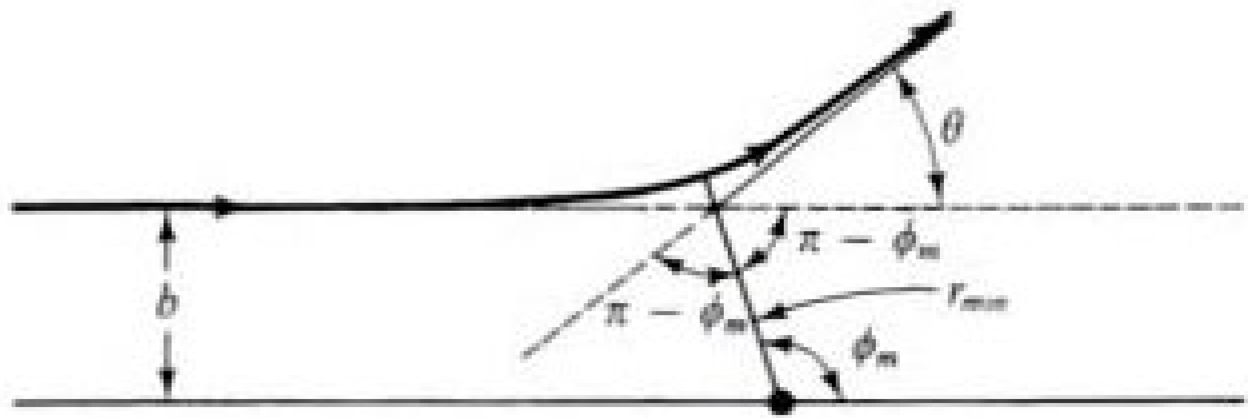
$$\Rightarrow \varphi_{\max}(b, E) = \varphi\left(r \to \infty\right) - \varphi\left(r = r_{\min}\right)$$

$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where :

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Relationship between ϕ_{max} and θ:



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

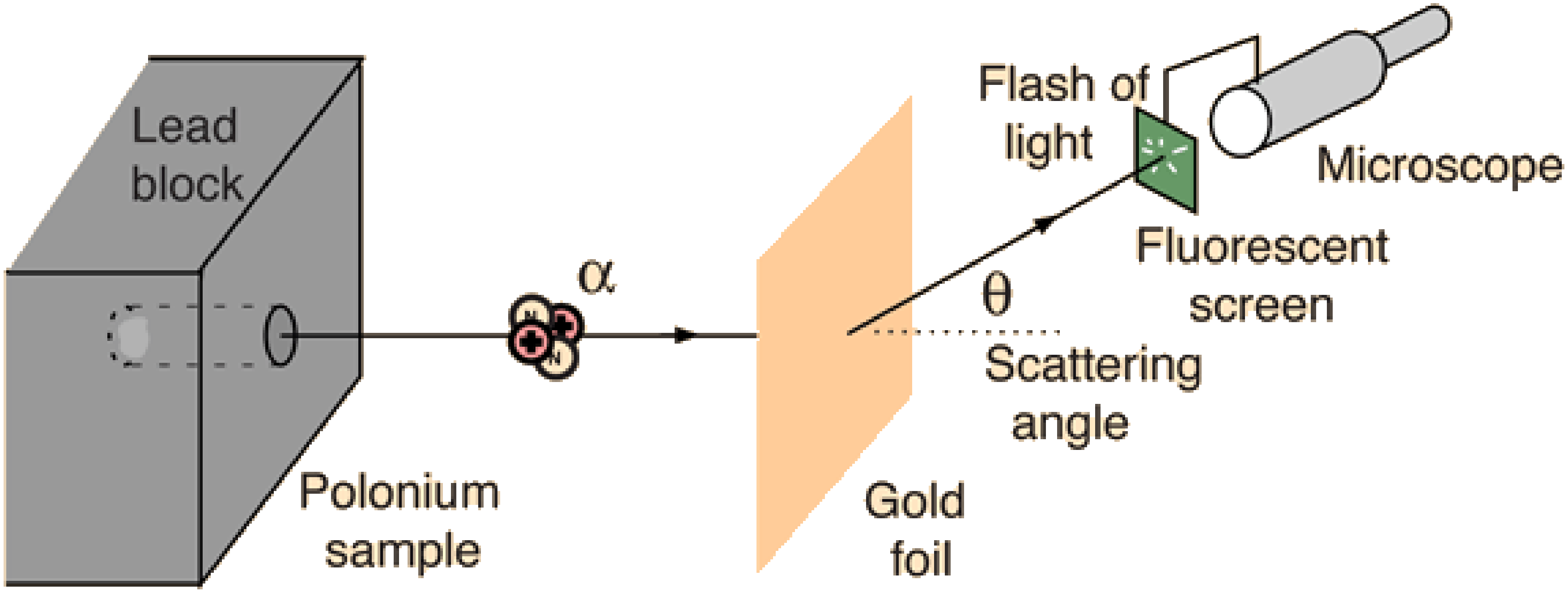
Using the diagram from your text, θ represents the scattering angle in the center of mass frame.

$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1 / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

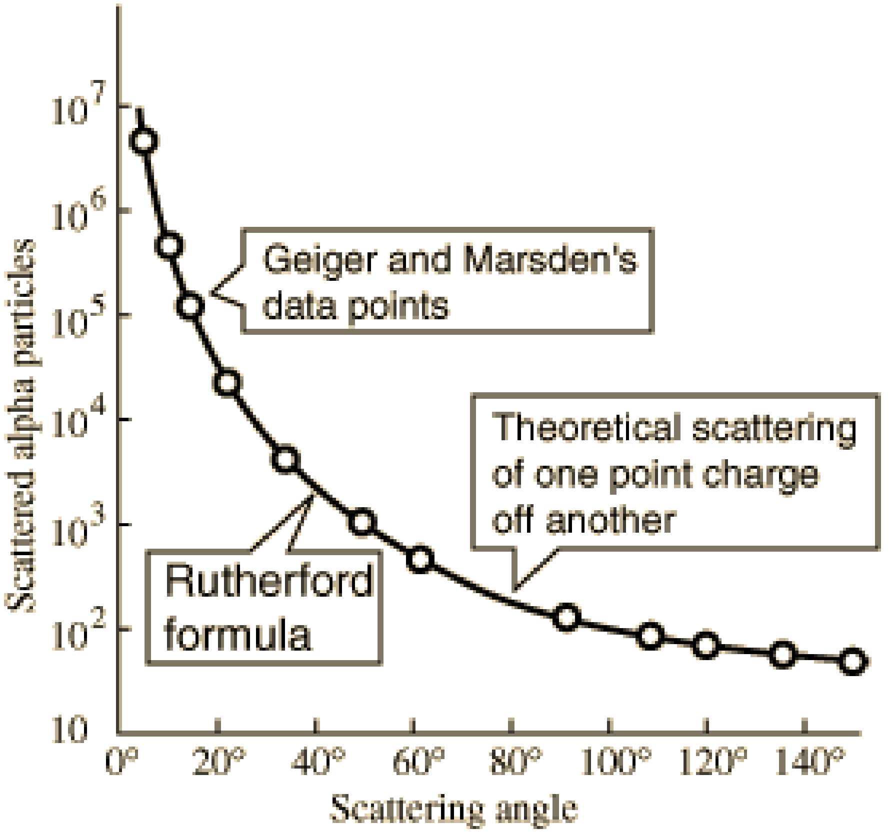
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\text{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\text{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

Question –

What do you think happens for $\theta \rightarrow 0$?

a. Big trouble; need to make sure experiment is designed to avoid that case.
b. No problem
   i. Physics is altered in that case and nothing explodes.
   ii. Rare event and rarely causes trouble.

Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\theta = -\pi + 2b\int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where $r_{\min}$ is found from

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

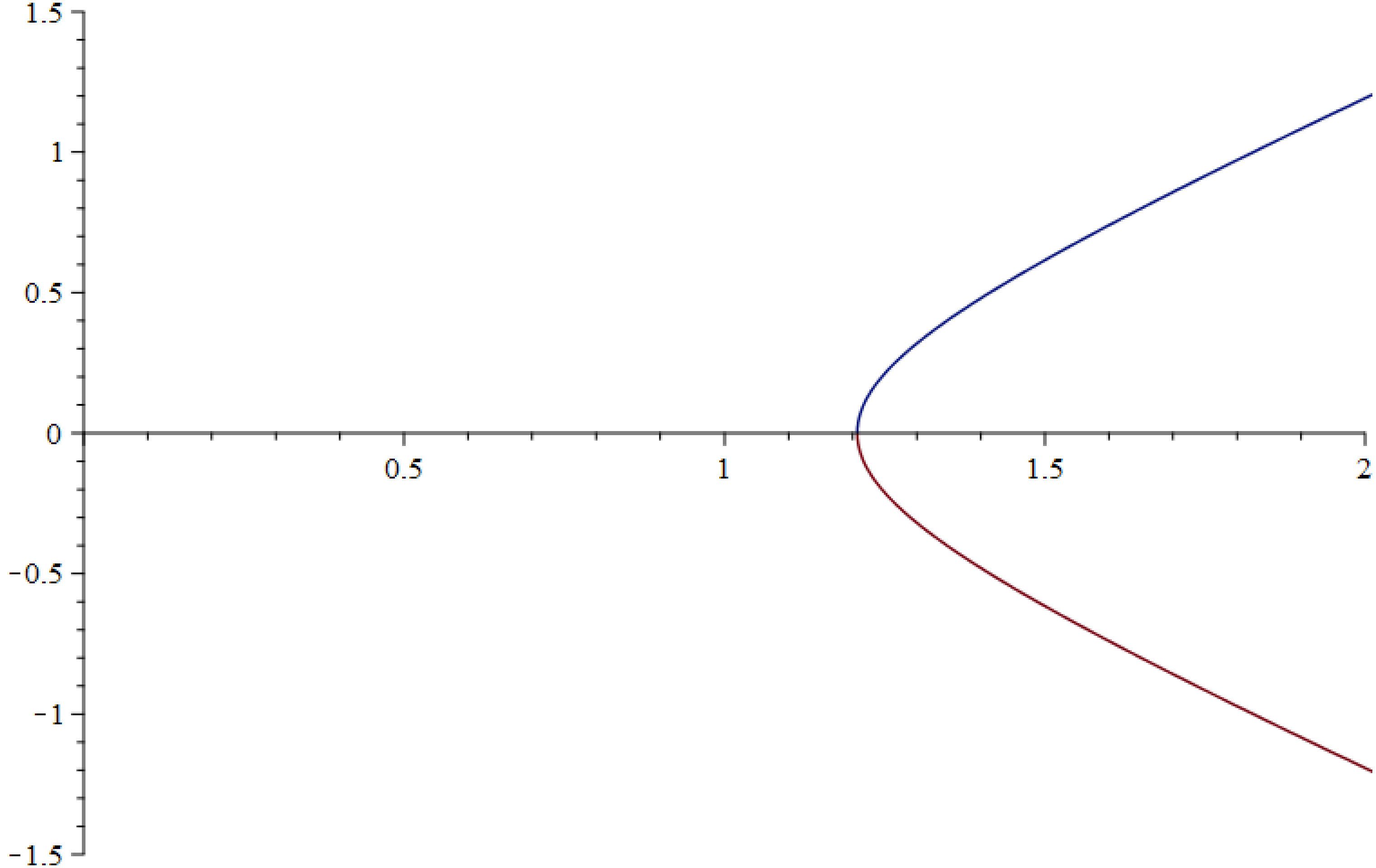
Digression– In general, it is possible to determine the trajectory *r(ϕ)* --

$$\varphi = \int_{r_{\min}}^{r} ds \left(\frac{b/s^2}{\sqrt{1 - \frac{b^2}{s^2} - \frac{V(s)}{E}}} \right)$$

$$\varphi = b \int_{1/r}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

For the Rutherford case --

$$\varphi = b \int_{1/r}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right) \quad \text{where } \frac{V(1/u)}{E} \equiv \kappa u$$



PHY 711 -- Assignment #3

Aug. 31, 2020

Read Chapter 1 in **Fetter & Walecka**.

1.In Lecture 3, we derived equations relating the laboratory scattering angle to the scattering angle in the center of mass reference frame. We also worked out the relationship between the differential scattering cross sections in the laboratory and center of mass frames. After you have convinced yourselves of the validity of those derivations, evaluate both the lab and center of mass scattering angles and the corresponding cross section factors for the following mass ratios

a. $m_1/m_2=1$
b. $m_1/m_2=1/10$
c. $m_1/m_2=10/1$

Transformation between center-of-mass and laboratory reference frames: (assuming that energy is conserved)

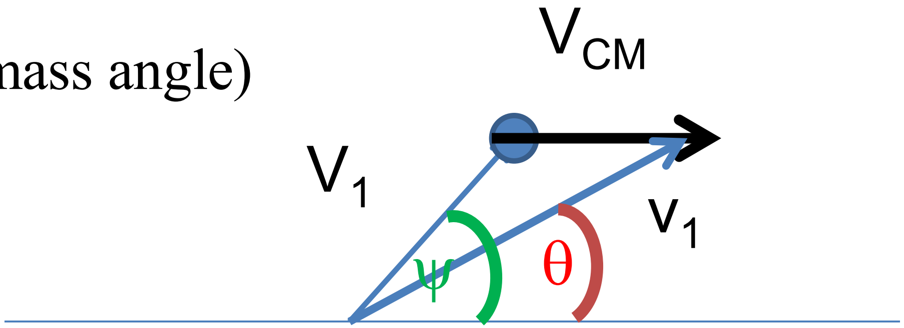
θ (lab angle) vs ψ (center of mass angle)

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$



Also: $$\cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2 \cos\psi + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames –

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right|=\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

For elastic scattering:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

$$\text{where:}\qquad \tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

Want to calculate

$$\frac{\left(\dfrac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)}{\left(\dfrac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)}=\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}=f(\psi)$$

where: $$\tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

$$\theta(\psi)=\arctan\left(\frac{\sin\psi}{\cos\psi+m_1/m_2}\right)$$

want to evaluate f for values of θ

$\Rightarrow$ parametric plot of $f(\psi)$ vs $\theta(\psi)$

Another example of a parametric plot

$x(t)$ vs $y(t)$

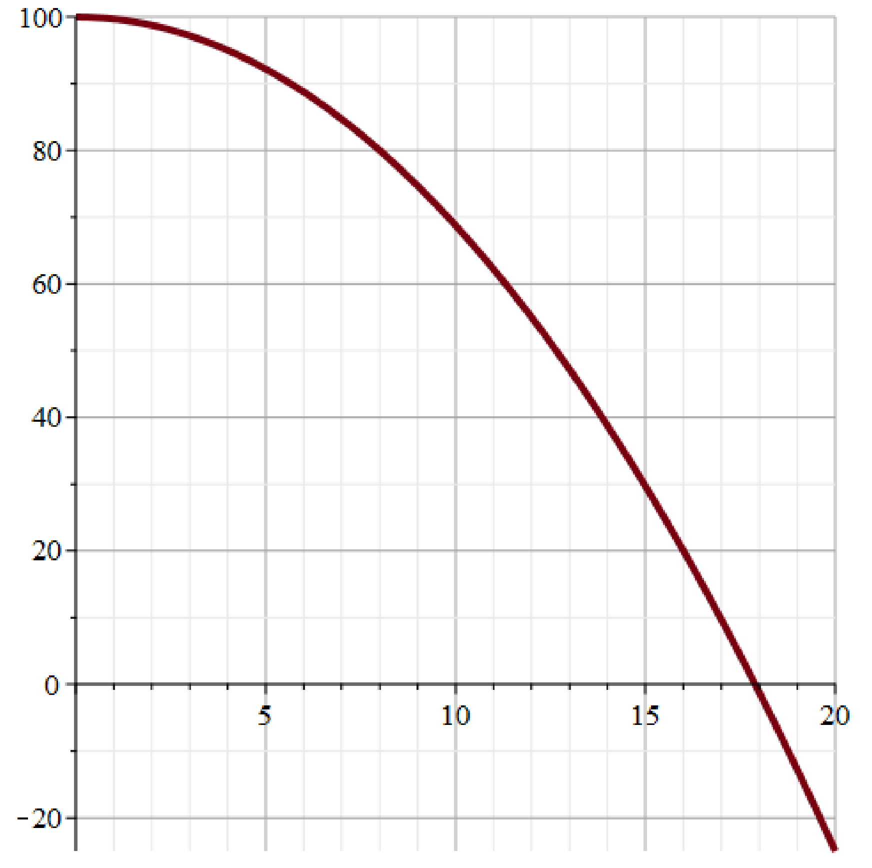
Syntax for parametric plot in maple

> $x := t \rightarrow 4 \cdot t; y := t \rightarrow 100 - \frac{1}{2} \cdot 10 \cdot t^2;$

$$x := t \mapsto 4 \cdot t$$

$$y := t \mapsto 100 - 5 \cdot t^2$$

> $plot([x(t), y(t), t = 0..5]);$



Syntax for parametric plot in mathematica

In[5]:=

```
ParametricPlot[{4*t, 50 - 2*t^2}, {t, 0, 5}]
```

Out[5]=