PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Discussion notes for Lecture 6

Physics analyzed in accelerated coordinate frames – Chap 2 F&W

1. Angular acceleration
2. Linear and angular acceleration
3. Foucault pendulum

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation		

Your questions –

From Tim

1. In slide 8 I don't know where the w in the first parentheses comes from on the third line.

From Nick

1. Can you explain the signs with the direction vectors for the final result [of the coordinate rotation]? Why do these point in the negative in the negative y and positive z directions?
2. What vector or direction is the [x unit vector] representing?
3. What happens to the term[involving the time rate of change of omega] when you are calculating gravity?

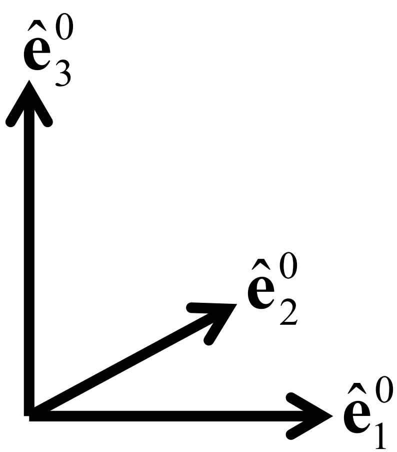
From Gao

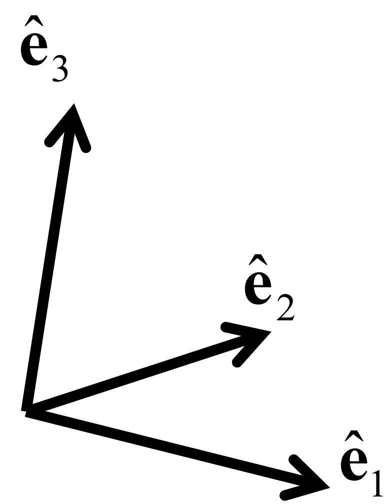
1. What is the meaning of “a” in the equations for accelerated reference frames.

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\left\{ \hat{\mathbf{e}}_i^0 \right\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system

$$\left\{ \hat{\mathbf{e}}_i \left(t \right) \right\}$$





Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{\mathbf{e}}_i^0$ an fixed coordinate system in 3 orthogonal directions

Denote by $\hat{\mathbf{e}}_i$ a moving coordinate system in 3 orthogonal directions

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{\mathbf{e}}_i^0 = \sum_{i=1}^{3} V_i \hat{\mathbf{e}}_i$$

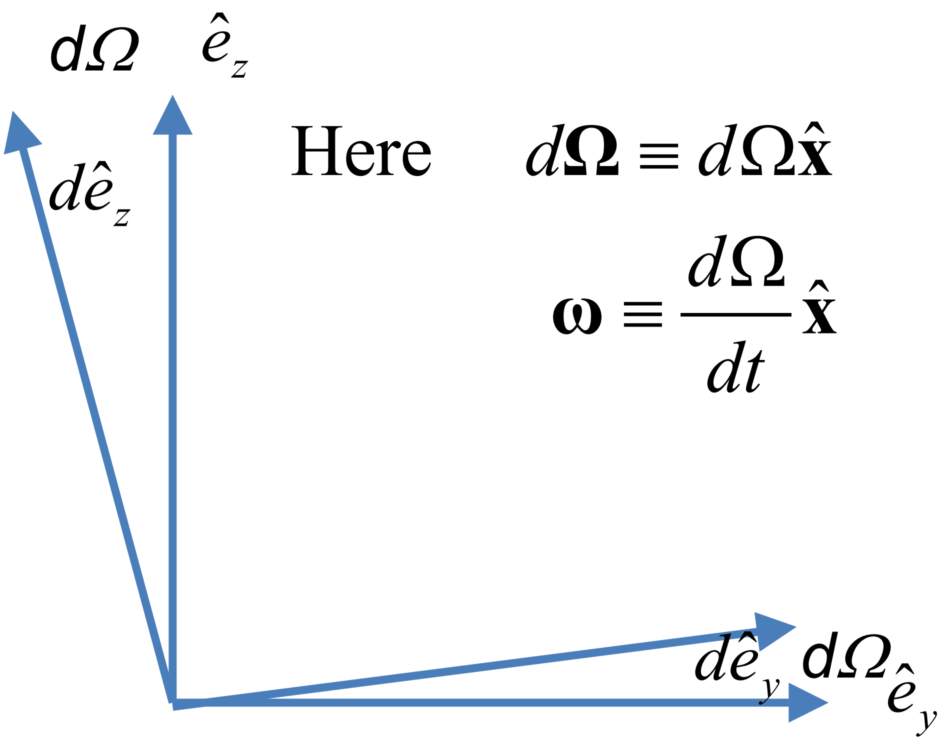
$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt} \hat{\mathbf{e}}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt} \hat{\mathbf{e}}_i + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

Define: $$\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt} \hat{\mathbf{e}}_i$$

This represents the time rate of change of V measured within the e frame.

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

Properties of the frame motion (rotation only):



Here $d\boldsymbol{\Omega} \equiv d\Omega\hat{\mathbf{x}}$

$$\boldsymbol{\omega} \equiv \frac{d\Omega}{dt}\hat{\mathbf{x}}$$

$$d\hat{e}_y = d\Omega\hat{e}_z$$

$$d\hat{e}_z = -d\Omega\hat{e}_y$$

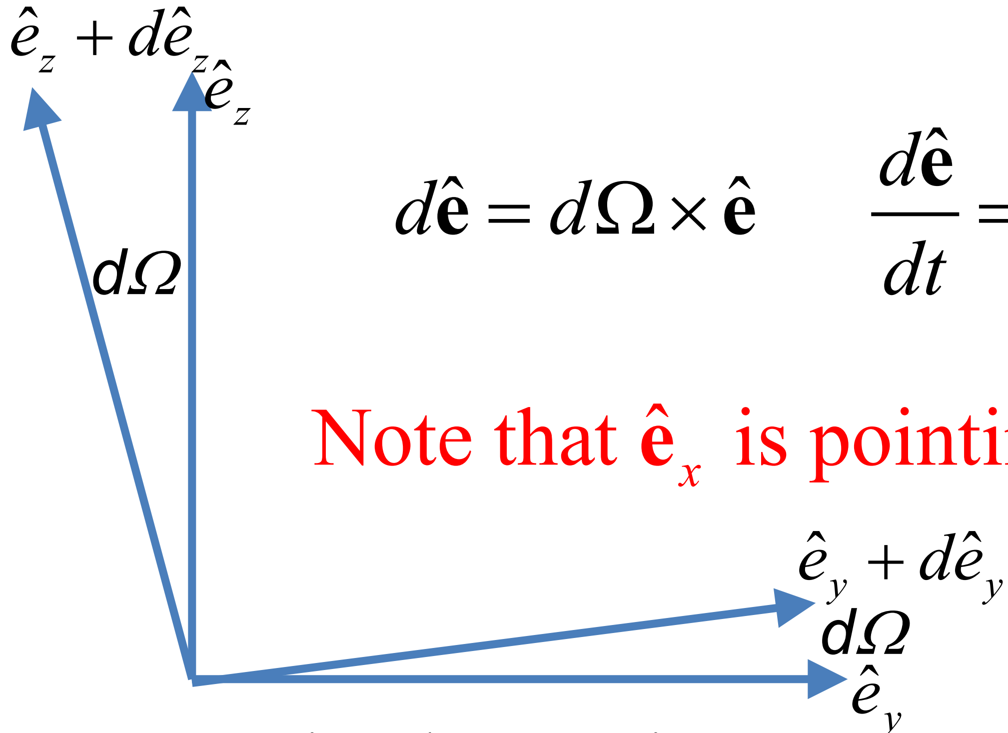
$$\Rightarrow d\hat{\mathbf{e}} = d\boldsymbol{\Omega} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\boldsymbol{\Omega}}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Note that the coordinate $\hat{\mathbf{e}}_x$ is pointing out of the screen.

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\boldsymbol{\Omega} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\boldsymbol{\Omega}}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Note that $\hat{\mathbf{e}}_x$ is pointing out of the screen.

rotation matrix

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

More details

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$e_y + de_y = \cos(d\Omega) e_y + \sin(d\Omega) e_z$$

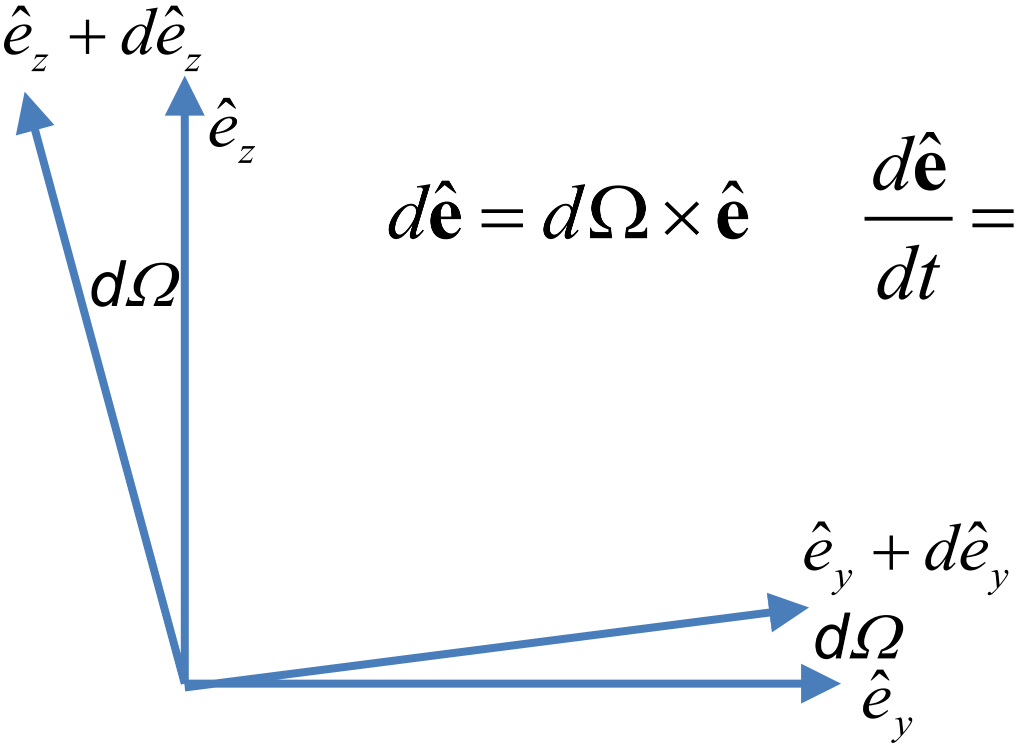
$$e_z + de_z = -\sin(d\Omega) e_y + \cos(d\Omega) e_z$$

Taylor's series

$$f(x_0 + dx) = f(x_0) + dx \left.\frac{df}{dx}\right|_{x_0} + \frac{1}{2}(dx)^2 \left.\frac{d^2 f}{dx^2}\right|_{x_0} + ...$$

$$\sin(dx) = dx - \frac{1}{6}(dx)^3 .... \qquad \cos(dx) = 1 - \frac{1}{2}(dx)^2 ....$$

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = d\Omega e_z \hat{\mathbf{y}} - d\Omega e_y \hat{\mathbf{z}} = d\Omega \hat{\mathbf{x}} \times \hat{\mathbf{e}}$$

Define axial vector $\mathbf{d\Omega} \equiv d\Omega \hat{\mathbf{x}}$

Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\mathbf{V}$$

Effects on 2nd time derivative -- acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{V} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{V}$$

Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration **a**

(Here we generalize previous case to add linear acceleration **a**.)

Newton's laws; Let **r** denote the position of particle of mass *m*:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = m\left(\mathbf{a} + \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\right) = \mathbf{F}_{ext}$$

Rearranging to find the effective acceleration within the non-inertial frame --

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\mathbf{a} - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Coriolis force

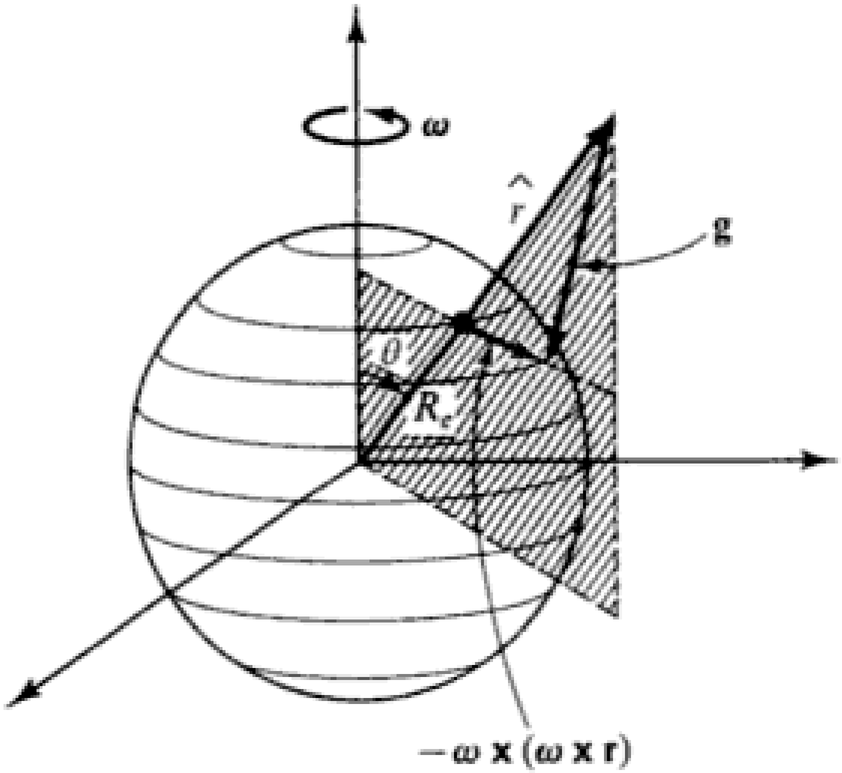
Centrifugal force

Have you every experienced any of these “fictitious” forces?

Examples –

1. Playing on a swing
2. Playing on a merry-go-round
3. Riding on a roller coaster
4. Sitting on the surface of the earth
5. Astronaut aboard the International Space Station

Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3\times 10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Earth's gravity

Support force

Main contributions:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

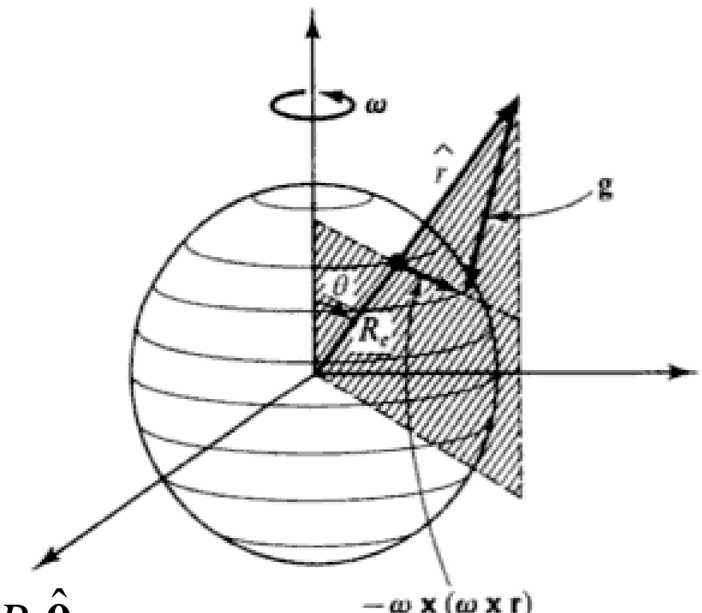
For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\frac{GM_e}{R_e^{\,2}} + \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$



9.80 m/s² 0.03 m/s²

Note that in the previous analysis we left out the term $-m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}$

Is this justified?

1. Yes
2. No

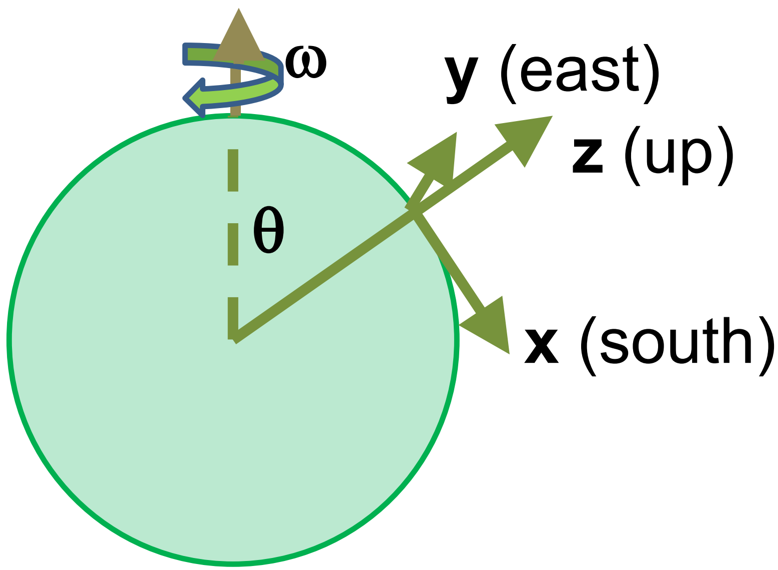
Foucault pendulum

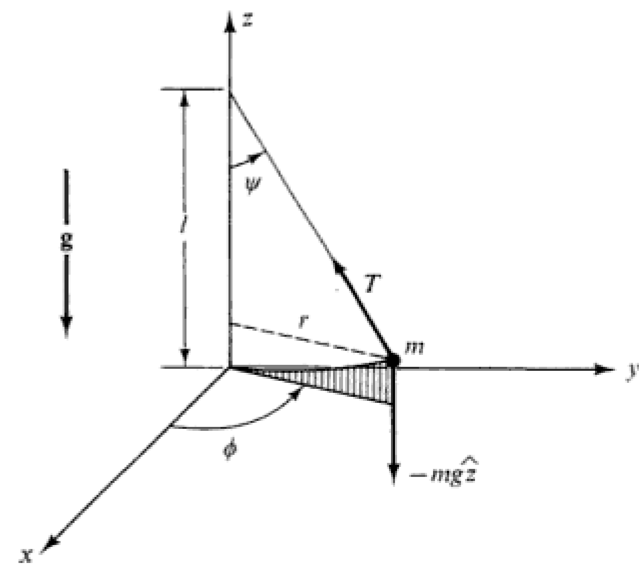
http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

Equation of motion on Earth's surface

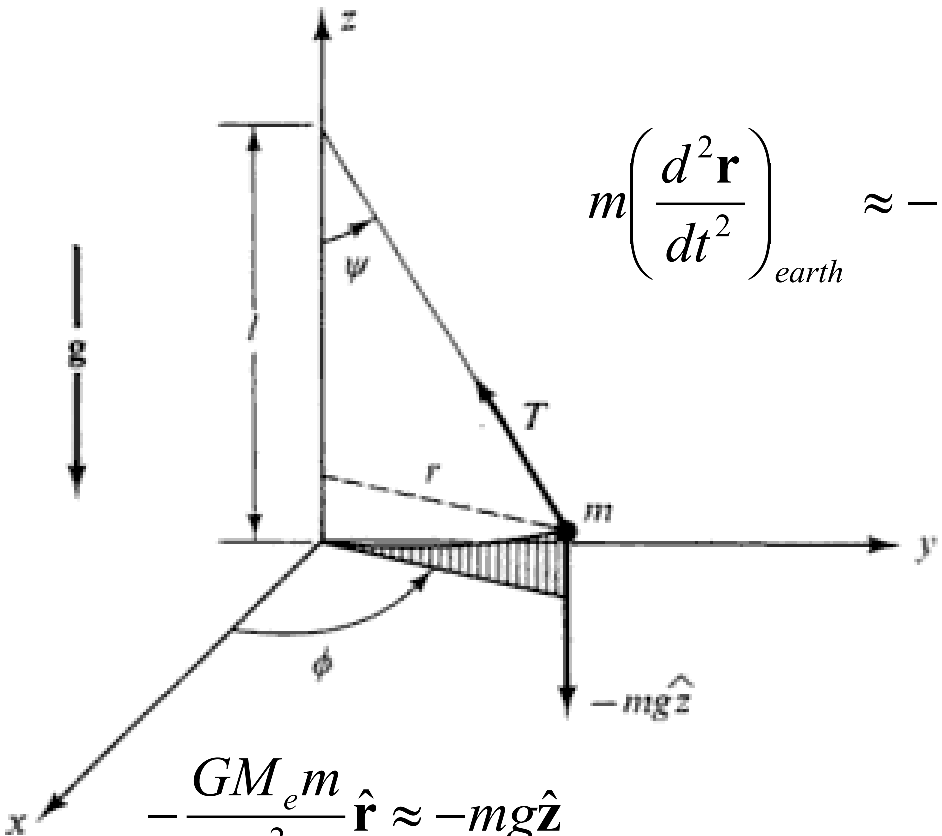
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T\sin\psi\cos\phi\hat{\mathbf{x}} - T\sin\psi\sin\phi\hat{\mathbf{y}} + T\cos\psi\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}} + \left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)\hat{\mathbf{y}} - \dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

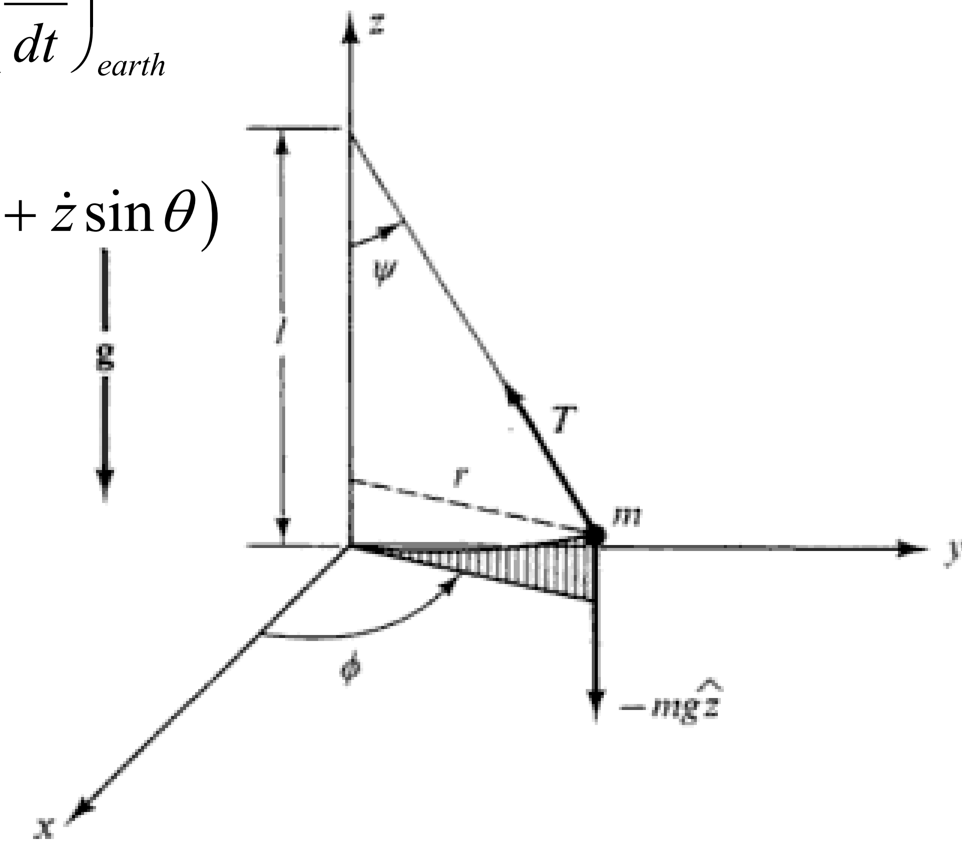
Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^{\ 2}}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x} \approx -T\sin\psi\cos\varphi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -T\sin\psi\sin\varphi - 2m\omega\left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)$$

$$m\ddot{z} \approx T\cos\psi - mg + 2m\omega\dot{y}\sin\theta$$



Further approximation :

$\psi << 1;\quad \ddot{z} \approx 0;\quad T \approx mg$

$$m\ddot{x} \approx -mg\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -mg\sin\psi\sin\phi - 2m\omega\dot{x}\cos\theta$$

Also note that :

$$x \approx \ell\sin\psi\cos\phi$$

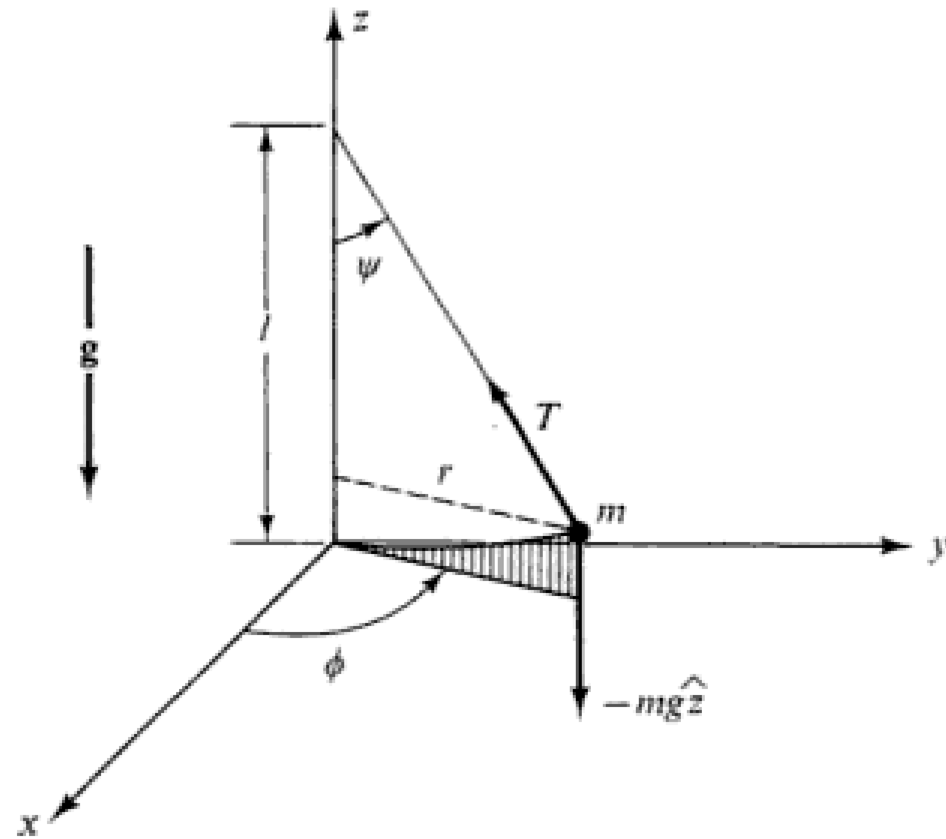
$$y \approx \ell\sin\psi\sin\phi$$

ℓ denotes the length of the rope/wire

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}$$

$$\ddot{y} \approx -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}$$



Try to find a solution of the form:

$$x(t) = Xe^{-iqt} \qquad y(t) = Ye^{-iqt}$$

Denote $\omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}}$$

Foucault pendulum continued – coupled equations:

Solution continued :

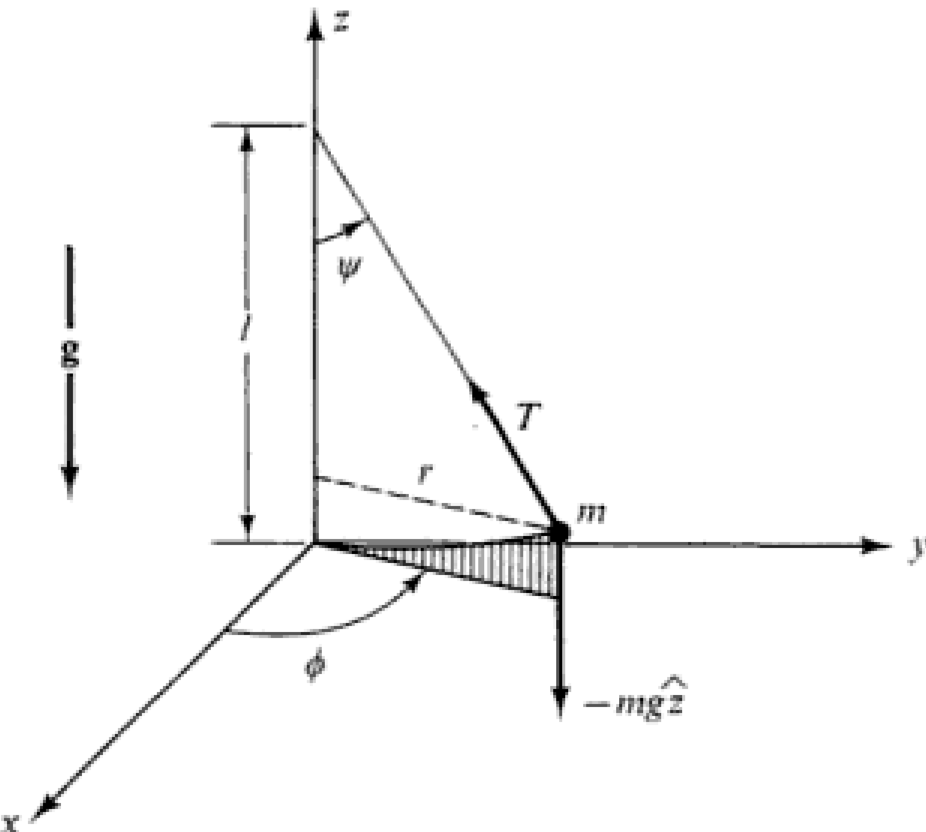
$$x(t) = Xe^{-iqt} \qquad y(t) = Ye^{-iqt}$$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions :

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}}$$

Amplitude relationship : $\quad X = iY$



General solution with complex amplitudes C and D :

$$x(t) = \mathrm{Re}\left\{ iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t} \right\}$$

$$y(t) = \mathrm{Re}\left\{ Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t} \right\}$$

General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}{}^2 + \frac{g}{\ell}} \approx \omega_{\perp} \pm \sqrt{\frac{g}{\ell}}$$

since $\quad \omega_{\perp} \approx 7\times 10^{-5} \cos\theta \;\; rad/s << \sqrt{\frac{g}{\ell}}$

Suppose: $\quad x(0) = X_0 \qquad y(0) = 0$

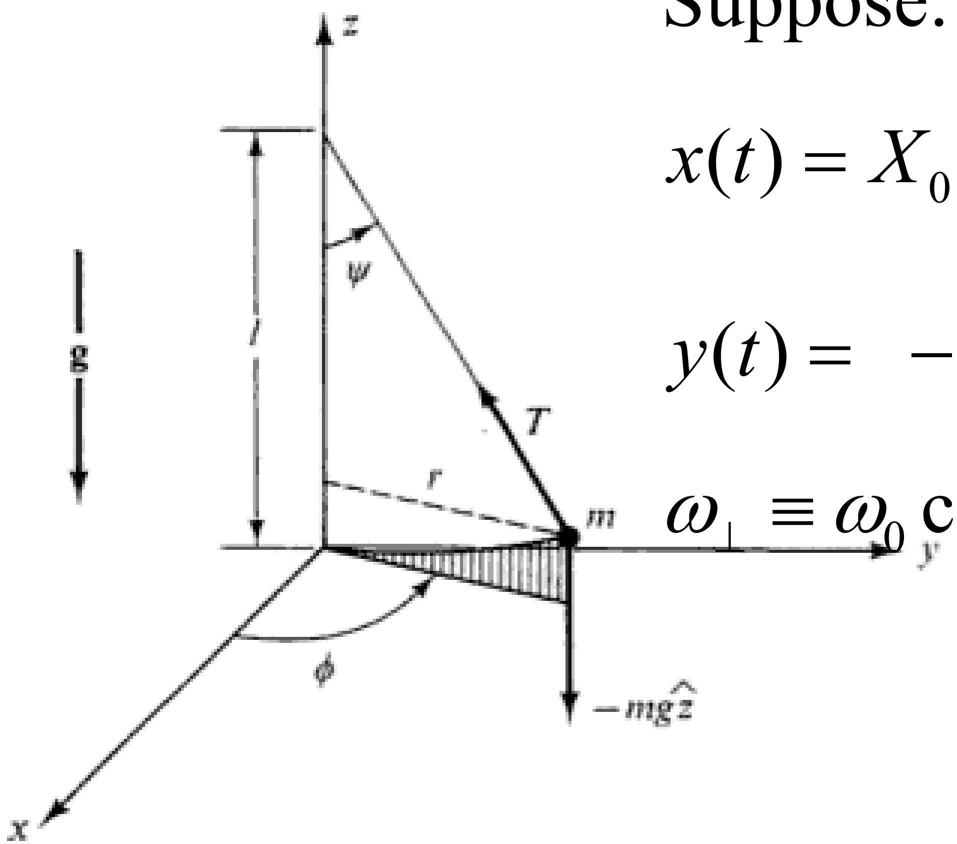
$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos\left(\omega_{\perp} t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin\left(\omega_{\perp} t\right)$$

Note that

$$\omega = \frac{2\pi}{24\cdot 3600 \text{ s}} = 7\times 10^{-5} \text{ rad/sec}$$

Summary of approximate solution for Foucault pendulum:

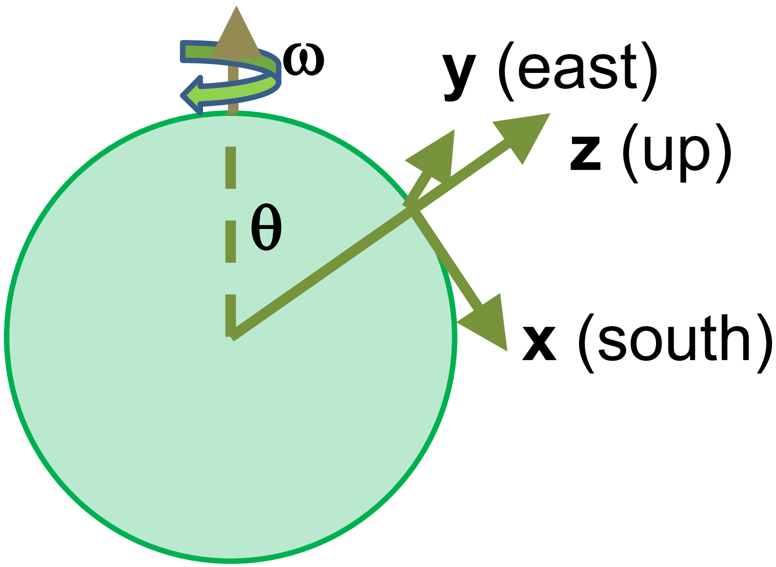


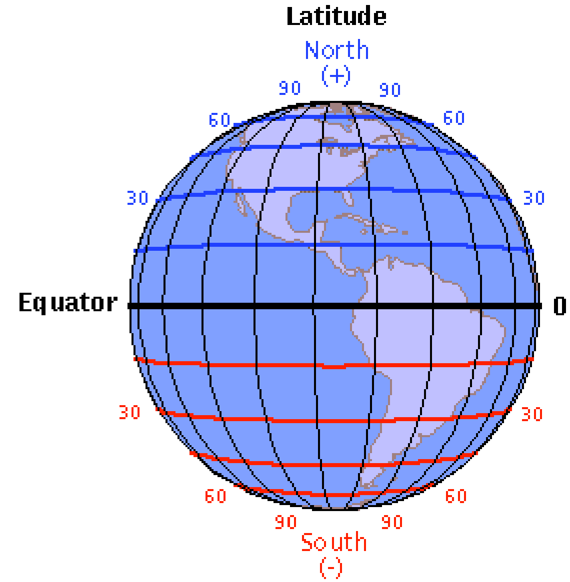
Suppose: $x(0) = X_0 \qquad y(0) = 0$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin\left(\omega_\perp t\right)$$

$$\omega_\perp \equiv \omega_0 \cos\theta$$





Microsoft Illustration

$$\omega_\perp \equiv \omega_0 \cos\theta$$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = \; -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin\left(\omega_\perp t\right)$$

Latitude and Longitude

https://www.latlong.net/

Latitude and Longitude Finder

Latitude and Longitude are the units that represent the *coordinates at geographic coordinate system*. To make a search, use the name of a place, city, state, or address, or click the location on the map to **find lat long coordinates**.

Place Name

Winston-Salem, NC, USA

Find

Add the country code for better results. Ex: London, UK

Latitude	Longitude
36.096260	-80.243736

Note that $\theta = 90^o$ - Latitude