PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Plan for Lecture 6: -- Chapter 2 of F & W

1. Physics analyzed in accelerated coordinate frames
   a. Linear acceleration
   b. Angular acceleration
   c. Foucault pendulum

In this lecture, we will briefly discuss the analysis of physics within accelerated reference frames as presented in Chapter 2 of your textbook.

Course schedule

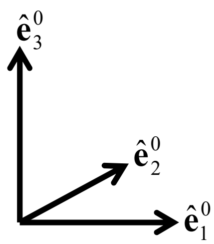
(Preliminary schedule -- subject to frequent adjustment.)

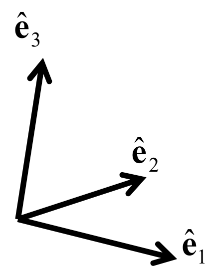
	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation		

We plan only one lecture on this topic, but it will come up again later in the course. On Wednesday, we will jump into the topic of calculus of variation in order to develop mathematical tools that form the backbone of the study of mechanics among other applications.

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\left\{\hat{\mathbf{e}}_i^0\right\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system $\left\{\hat{\mathbf{e}}_i(t)\right\}$





By "inertial" frame of reference, we mean a reference frame that is either stationary or is moving at constant velocity. A non-inertial frame of reference is the opposite.

Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{\mathbf{e}}_i^0$ an fixed coordinate system in 3 orthogonal directions

Denote by $\hat{\mathbf{e}}_i$ a moving coordinate system in 3 orthogonal directions

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{\mathbf{e}}_i^0 = \sum_{i=1}^{3} V_i \hat{\mathbf{e}}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt} \hat{\mathbf{e}}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt} \hat{\mathbf{e}}_i + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

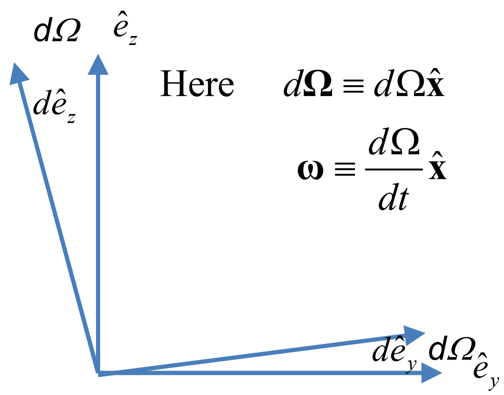
Define: $$\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt} \hat{\mathbf{e}}_i$$

This represents the time rate of change of V measured within the e frame.

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

The first question is how analyze quantities such as a vector V in the two frames of reference. The vector V may vary in time and its components

Properties of the frame motion (rotation only):



Here $d\mathbf{\Omega} \equiv d\Omega\hat{\mathbf{x}}$

$$\boldsymbol{\omega} \equiv \frac{d\Omega}{dt}\hat{\mathbf{x}}$$

$$d\hat{e}_y = d\Omega\hat{e}_z$$

$$d\hat{e}_z = -d\Omega\hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}}$$

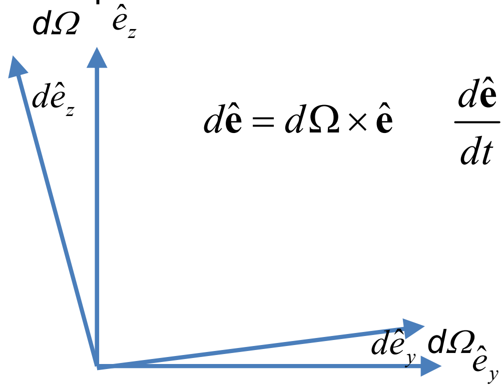
$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Note that the coordinate $\hat{\mathbf{e}}_x$ is pointing out of the screen.

Here omega is a vector whose magnitude is the time rate of change of rotation of the coordinate frame and whose direction is pointing along the axis of rotation (in this case the x-axis.

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

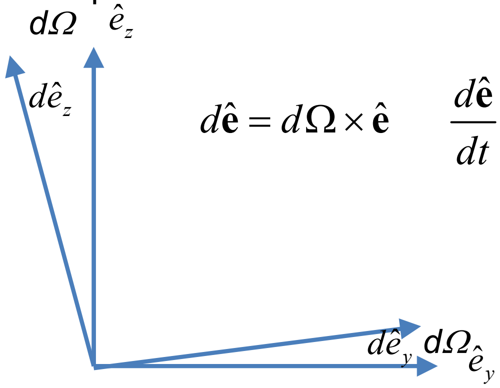
Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

Here we “derivate” the cross product relation using the expression for finite rotation and then expanding to infinitesimal angle dOmega.

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = -d\Omega e_z \hat{\mathbf{e}}_y + d\Omega e_y \hat{\mathbf{e}}_z = d\Omega \hat{\mathbf{x}} \times \hat{\mathbf{e}}$$

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Summary of previous results.

Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{V} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{V}$$

Having established how to analyze the first time derivative, we apply the time derivative to the result in order to analyze the second time derivative (acceleration).

Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration **a**

Newton's laws; Let **r** denote the position of particle of mass *m*:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = m\left(\mathbf{a} + \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\right) = \mathbf{F}_{ext}$$

Rearranging to find the effective acceleration within the non-inertial frame --

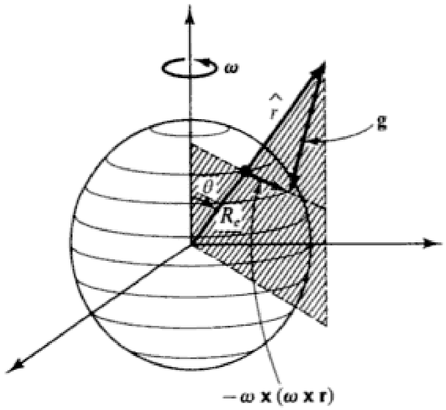
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\mathbf{a} - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Coriolis force

Centrifugal force

The highlighted terms are often called "fictictious" forces.

Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3\times10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Earth's gravity

Support force

Main contributions:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Considering now our case, standing or sitting on the surface of the rotating earth (neglecting other details like orbiting the sun). F' is the support force meaning the floor on which we are standing/sitting and its support due to earth's crust.

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

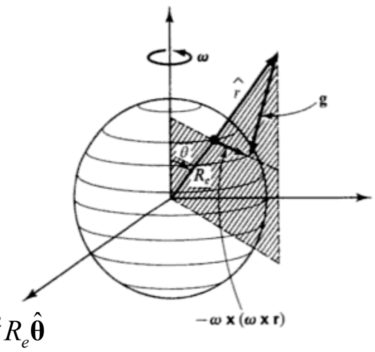
For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$



0.03 m/s²

9.80 m/s²

Some details.

Foucault pendulum http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

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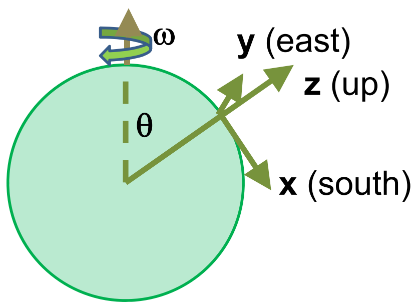
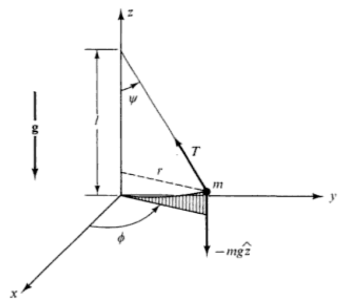
A very interesting example designed to show the effects of the earth's rotation.

Equation of motion on Earth's surface

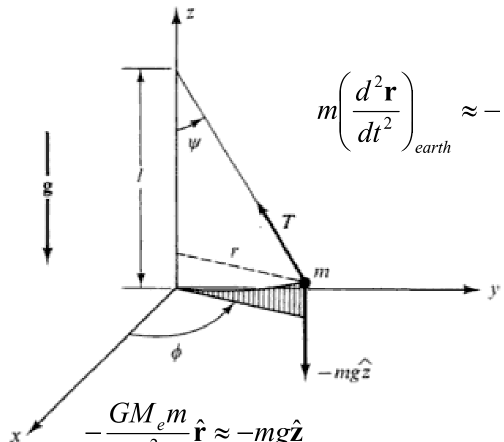
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^{\,2}}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T\sin\psi\cos\phi\hat{\mathbf{x}} - T\sin\psi\sin\phi\hat{\mathbf{y}} + T\cos\psi\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}} + \left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)\hat{\mathbf{y}} - \dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

Analyzing the pendulum motion in the given geometry.

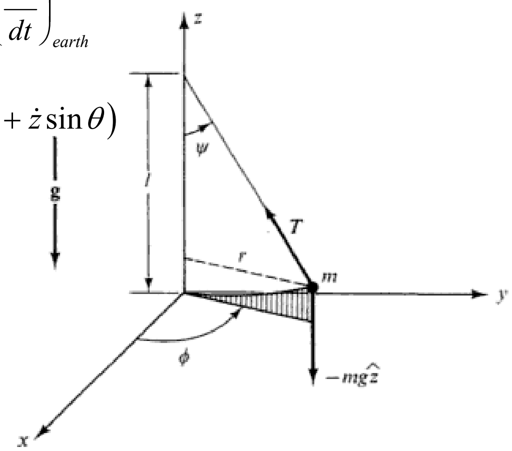
Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x} \approx -T\sin\psi\cos\varphi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -T\sin\psi\sin\varphi - 2m\omega\left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)$$

$$m\ddot{z} \approx T\cos\psi - mg + 2m\omega\dot{y}\sin\theta$$



Further approximation :

$\psi << 1;\quad \ddot{z} \approx 0;\quad T \approx mg$

$$m\ddot{x} \approx -mg\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -mg\sin\psi\sin\phi - 2m\omega\dot{x}\cos\theta$$

Also note that :

$$x \approx \ell\sin\psi\cos\phi$$

$$y \approx \ell\sin\psi\sin\phi$$

Simplifying the equations for the dominant terms.

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}$$

$$\ddot{y} \approx -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}$$

Try to find a solution of the form:

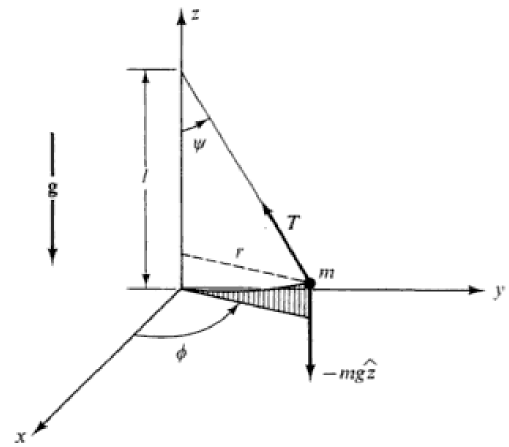
$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

Denote $\omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non-trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^2 + \frac{g}{\ell}}$$



More details

Foucault pendulum continued – coupled equations:

Solution continued :

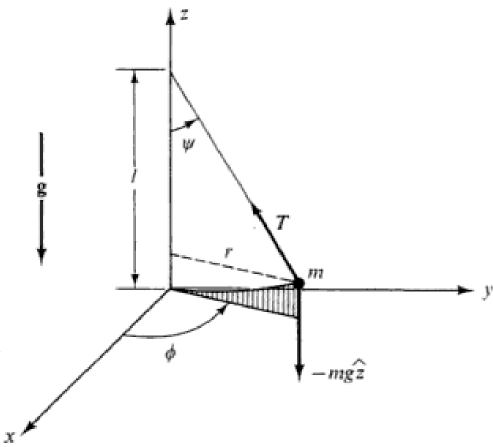
$$x(t) = Xe^{-iqt} \qquad y(t) = Ye^{-iqt}$$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions :

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}}$$

Amplitude relationship : $X = iY$



General solution with complex amplitudes C and D :

$$x(t) = \text{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \text{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

Solving the differential equations.

General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{ iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t} \right\}$$

$$y(t) = \mathrm{Re}\left\{ Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t} \right\}$$

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}{}^{2} + \tfrac{g}{\ell}} \approx \omega_{\perp} \pm \sqrt{\tfrac{g}{\ell}}$$

since $\quad \omega_{\perp} \approx 7\times10^{-5}\cos\theta \;\; rad/s << \sqrt{\tfrac{g}{\ell}}$

Suppose: $\quad x(0) = X_0 \qquad y(0) = 0$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\cos(\omega_{\perp} t)$$

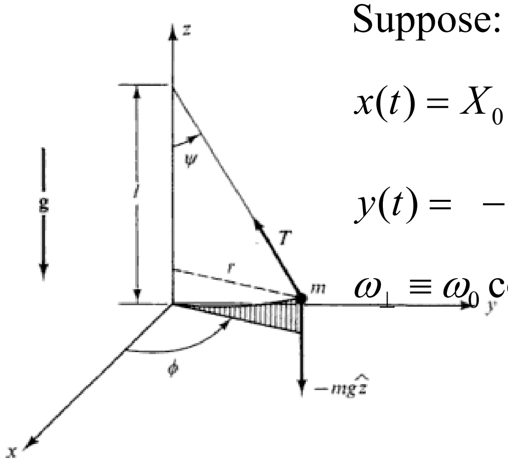
$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\sin(\omega_{\perp} t)$$

Note that

$$\omega = \frac{2\pi}{24\cdot 3600 \text{ s}} = 7\times10^{-5} \text{ rad/sec}$$

More details and approximations.

Summary of approximate solution for Foucault pendulum:



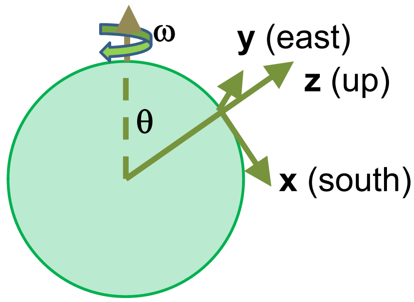
Suppose: $x(0) = X_0 \qquad y(0) = 0$

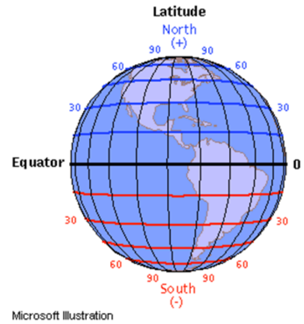
$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin\left(\omega_\perp t\right)$$

$$\omega_\perp \equiv \omega_0 \cos\theta$$

Summary





$$\omega_\perp \equiv \omega_0 \cos\theta$$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = \; -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin\left(\omega_\perp t\right)$$

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It is interesting to estimate the effects in different parts of the globe. Note that theta is defined for the polar angle, while if you look up your latitude you will find 90-theta.