PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin103

Plan for Lecture 7 -- Chapter 3.17 of F&W

Introduction to the calculus of variations

1. Mathematical construction
2. Practical use
3. Examples

The topic of “calculus of variation” is covered in Chapter 3, Section 17 of your textbook. We will study the mathematical formalism first before showing how it is useful for studying mechanical systems.

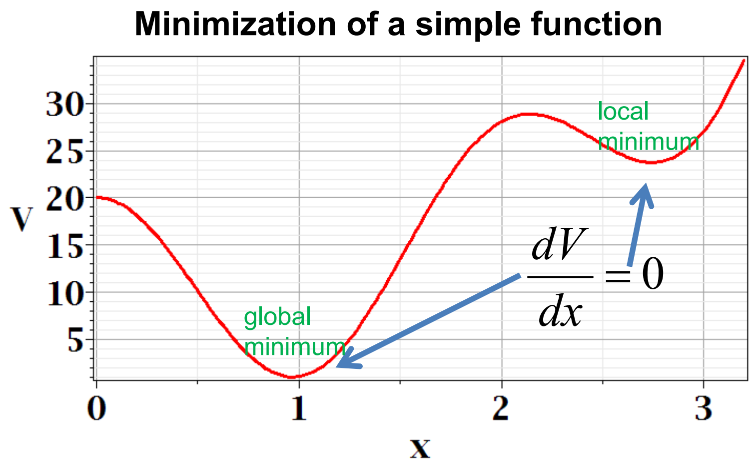
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020

There is a short problem on this subject that will be do on Friday.

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

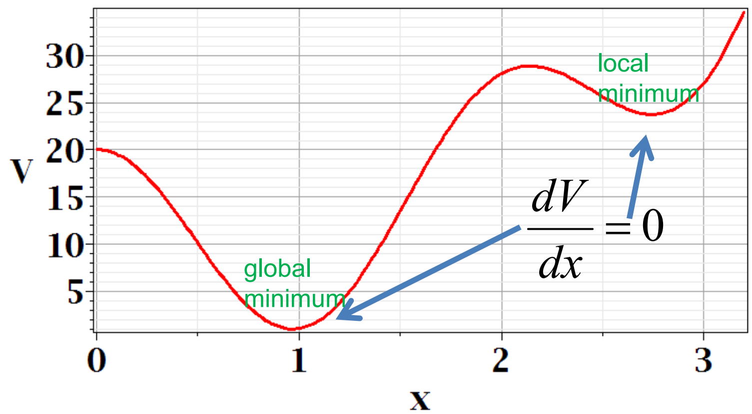


First we should review the notion of a minimum in a continuous function. Here is a plot of V(x) showing two different minima at two different points x.

Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition: $\frac{dV}{dx} = 0$



We see from this plot that a conduction for a function to have a minimum at a point is that its derivative is zero at that point. You see in this example another point where dV/dx, but there is not a minimum. So we say the dV/dx is a necessary but not sufficient condition on having a minimum.

Functional minimization

Consider a family of functions $y(x)$, with fixed end points

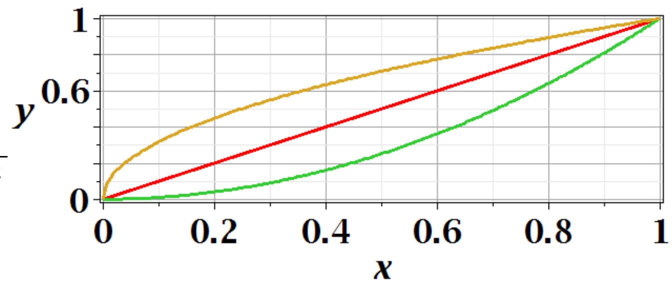
$y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



The calculus of variation also searches for minima, but instead of finding a point where a function has a minimum, we search for a functional form that minimizes an integral.

Difference between minimization of a function V(x) and the minimization in the calculus of variation.

Minimization of a function

➔Know V(x) ➔Find x_0 such that $V(x_0)$ is a minimum.

Calculus of variation

For $x_i \leq x \leq x_f$ want to find a function $y(x)$ that minimizes an integral that depends on $y(x)$.

Comparison

Functional minimization

Consider a family of functions $y(x)$, with fixed end points

$y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

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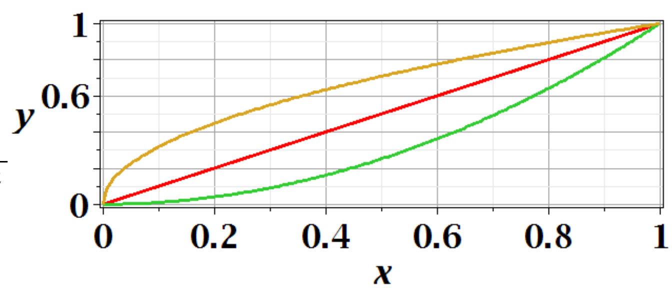
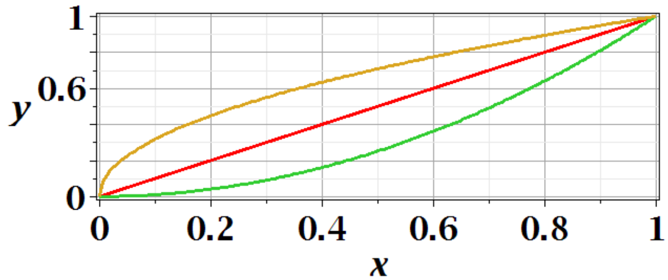


The calculus of variation involves consideration of a function of a function. Here we use L to denote such a function.

Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions :

$$y_1(x) = \sqrt{x} \qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$$

$$y_2(x) = x \qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$$

$$y_2(x) = x^2 \qquad L = \int_0^1 \sqrt{1+4x^2}\, dx = 1.4789$$

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For this example we can write the distance along a curve between two points x=0,y=0 and x=1,y=1 as a normal integral over x as shown.

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$

Necessary condition : $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y} \delta\left(\frac{dy}{dx}\right) \right] \right] dx.$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta\left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

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Using calculus to simplify the integral.

"Some" derivations --

Consider term

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx:$$

If $y(x)$ is a well-defined function, then $\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx=\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\frac{d}{dx}\delta y\right]dx$$

$$=\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\delta y\right]dx$$

Some details.

“Some” derivations (continued)--

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\qquad 0 \qquad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Euler-Lagrange equation:

$$\Rightarrow\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0 \quad \text{for all } x_i\le x\le x_f$$

Example: End points -- $y(0)=0; y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

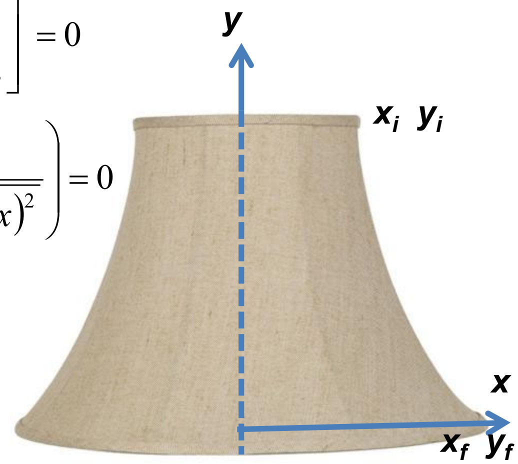
Your homework problem is very similar to this.

Example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$



Here is another example of the use of calculus of variation.

$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}=K_1$$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

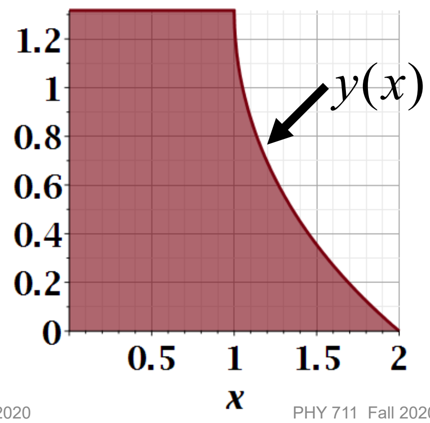
$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

After these steps, the solution is found up to some constants.

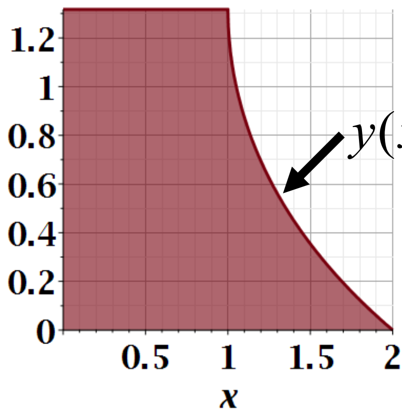
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1}\right)$$

Suppose $K_1 = 1$ and $K_2 = 2 + \sqrt{3}$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x + \sqrt{x^2 - 1}}\right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Evaluating results for particular boundary values.

Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0)=0$ and $y(1)=1$

that minimize the integral:

$$I=\int_0^1\left(\left(\frac{dy}{dx}\right)^2-ay^2\right)dx \quad \text{for constant } a>0$$

Euler-Lagrange equation:

$$\frac{d^2y}{dx^2}+ay=0$$

$$\Rightarrow y(x)=\frac{\sin\left(\sqrt{a}x\right)}{\sin\left(\sqrt{a}\right)}$$

Another example.

Review: for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

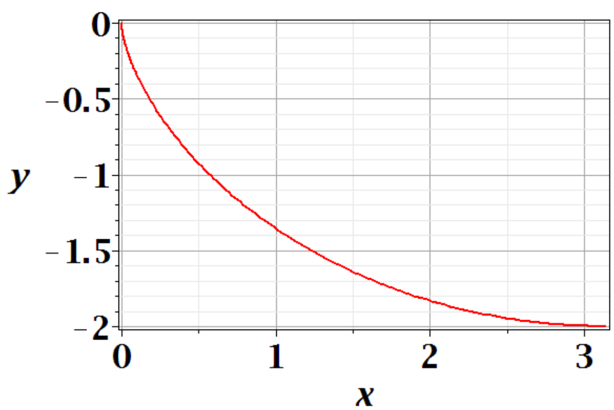
$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

Summary and extension.

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight mg travels frictionlessly down a path of shape $y(x)$. What is the shape of the path $y(x)$ that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

Prelude to what we will cover next time.

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\, dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Some details.

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

More details.

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a \qquad \text{Let} \quad y=-2a\sin^2\tfrac{\theta}{2}=a(\cos\theta-1)$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])