PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Discussion on Lecture 8 – Chap. 3 F & W

Calculus of variation

1. Brachistochrone problem
2. Calculus of variation with constraints
3. Application to classical mechanics

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Your questions

From Nick –

1. Can you go over the directions for the next assignment? I'm a bit confused as to what you're asking for in parts a,b.
2. Can you discuss a little more the setup for the T= integral equation at the top of slide 7? Why are we integrating ds/v?
3. Also, in the development of the alternative Euler-Lagrange, we get this relationship: $\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)$ Please explain.

From Tim –

1. On slide 8 you give the quantity inside the brackets as equal to K = 2a. Did you just do this to make it easier, because in the previous process it would have turned out to be 1/k^2.

From Gao –

1. About lecture #8, what does W=E+λL stand for (15th page of note)? Thank you.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020

PHY 711 – Assignment #6

September 7, 2020

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Evaluate
   $$\frac{\partial}{\partial q}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q}.$$

   (b) Evaluate
   $$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}}.$$

   (c) Evaluate
   $$\frac{df}{dt}.$$

   (d) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t^2, \quad \text{where} \quad q(t) = \mathrm{e}^{-t/\tau}.$$
   Here τ is a constant. Evaluate df/dt using the expression you just derived. Now find the expression for f as an explicit function of t ($f(t)$) and take its time derivative directly to check your previous results.

Your question -- please explain HW 6 a little better.

Suppose that we have a generalized coordinate $q(t)$ that varies with time t and a function that has the dependences $f\left(q(t),\frac{dq}{dt}(t);t\right)$. As an example, suppose that $q(t)=e^{-t/\tau}$ and $f\left(q(t),\dot{q}(t);t\right)=q\dot{q}^2t^2$ Evaluate $\frac{df}{dt}$ in two different ways for (d) and (c). You should get the same answer. For parts (a) and (b) you are asked to take two derivatives in different orders. The results may be surprising.

Review : for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

Your question – Also, in the development of the alternative Euler-Lagrange, we get this relationship:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)$$

Please explain.

Suppose that $y(x)$ is the function that extremizes the integral

$\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$ We then "derived" the Euler-Lagrange relation

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \Rightarrow \quad \left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} = \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right]$$

A few more steps --

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

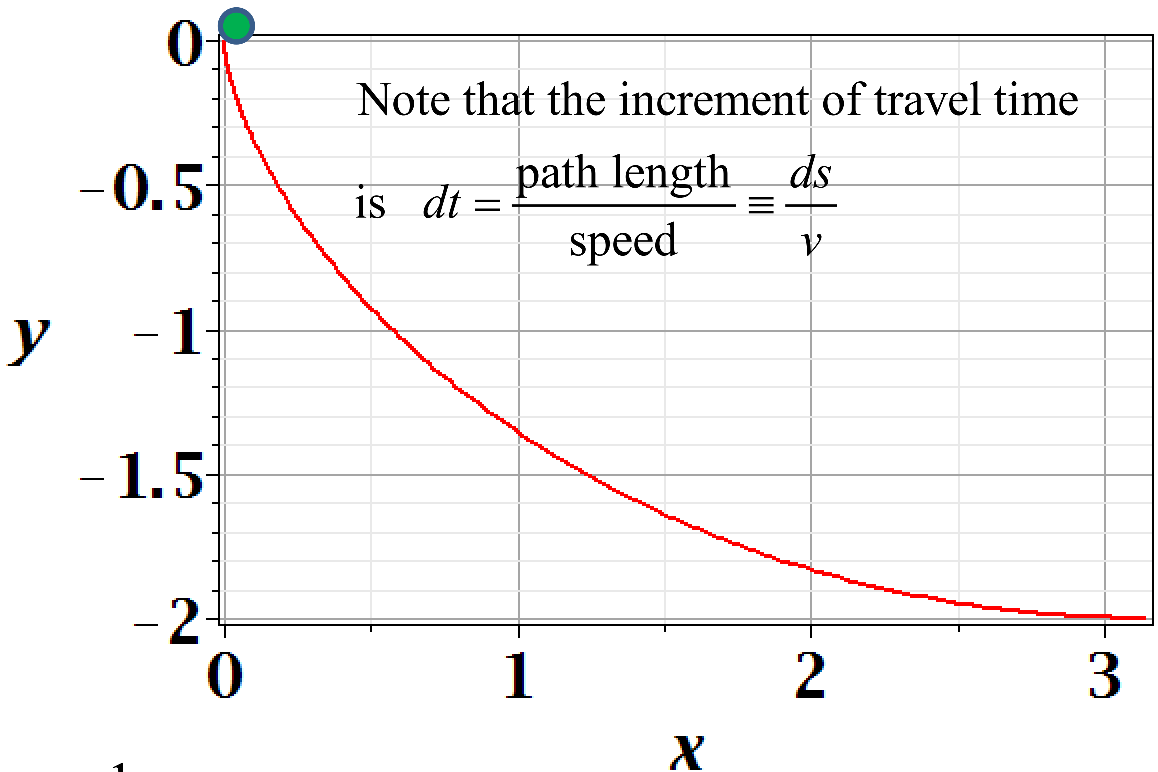
$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html

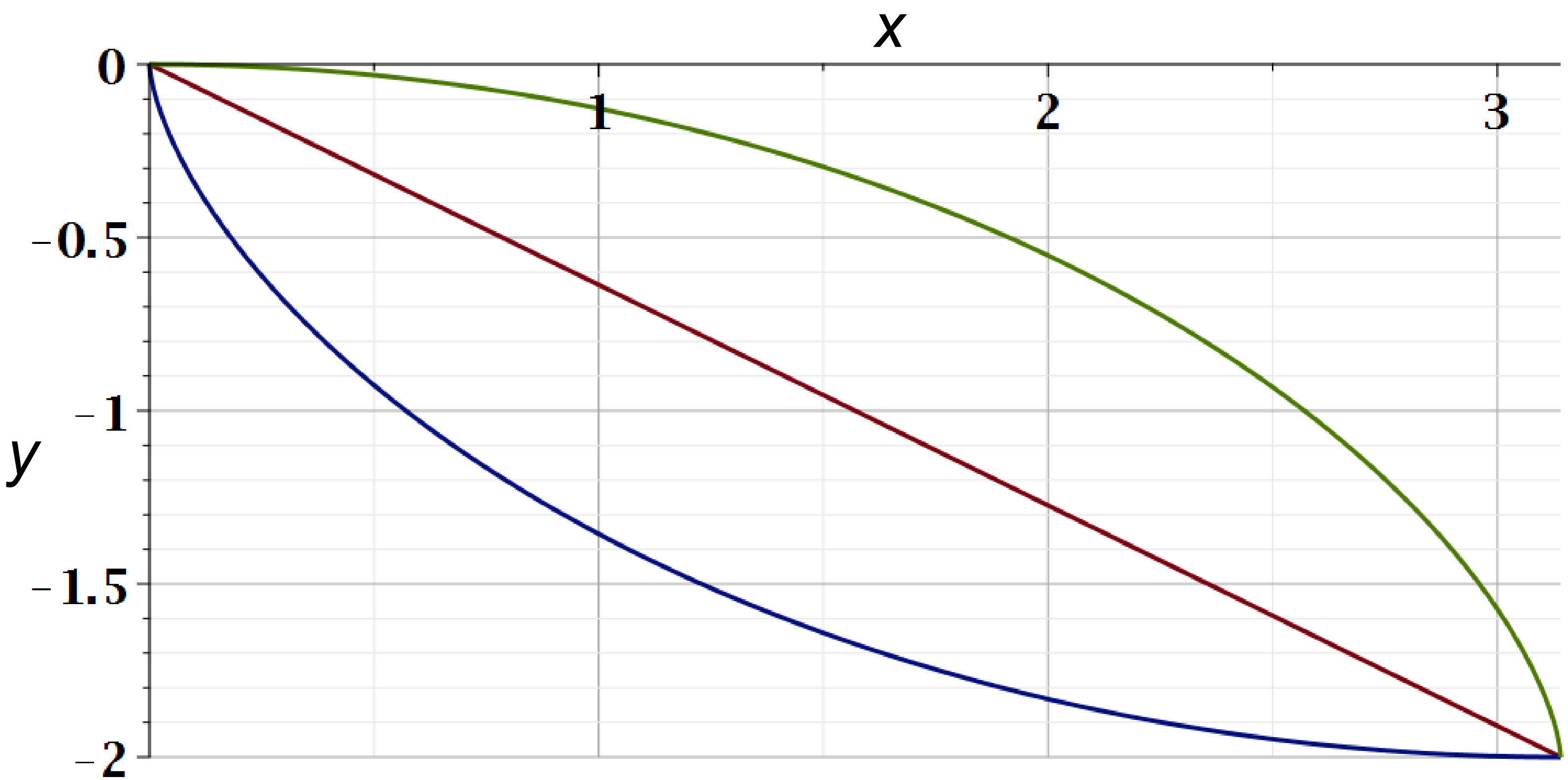


A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2* ?

$$E = \frac{1}{2}mv^2 + mgy$$

With the choice of initial conditions, $E = 0$

Vote for your favorite path



Which gives the shortest time?

a. Green
b. Red
c. Blue

$$T=\int_{x_i y_i}^{x_f y_f}\frac{ds}{v}=\int_{x_i}^{x_f}\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx \quad \text{because} \quad \tfrac{1}{2}mv^2=-mgy$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}}-\frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

Question – why this choice? Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

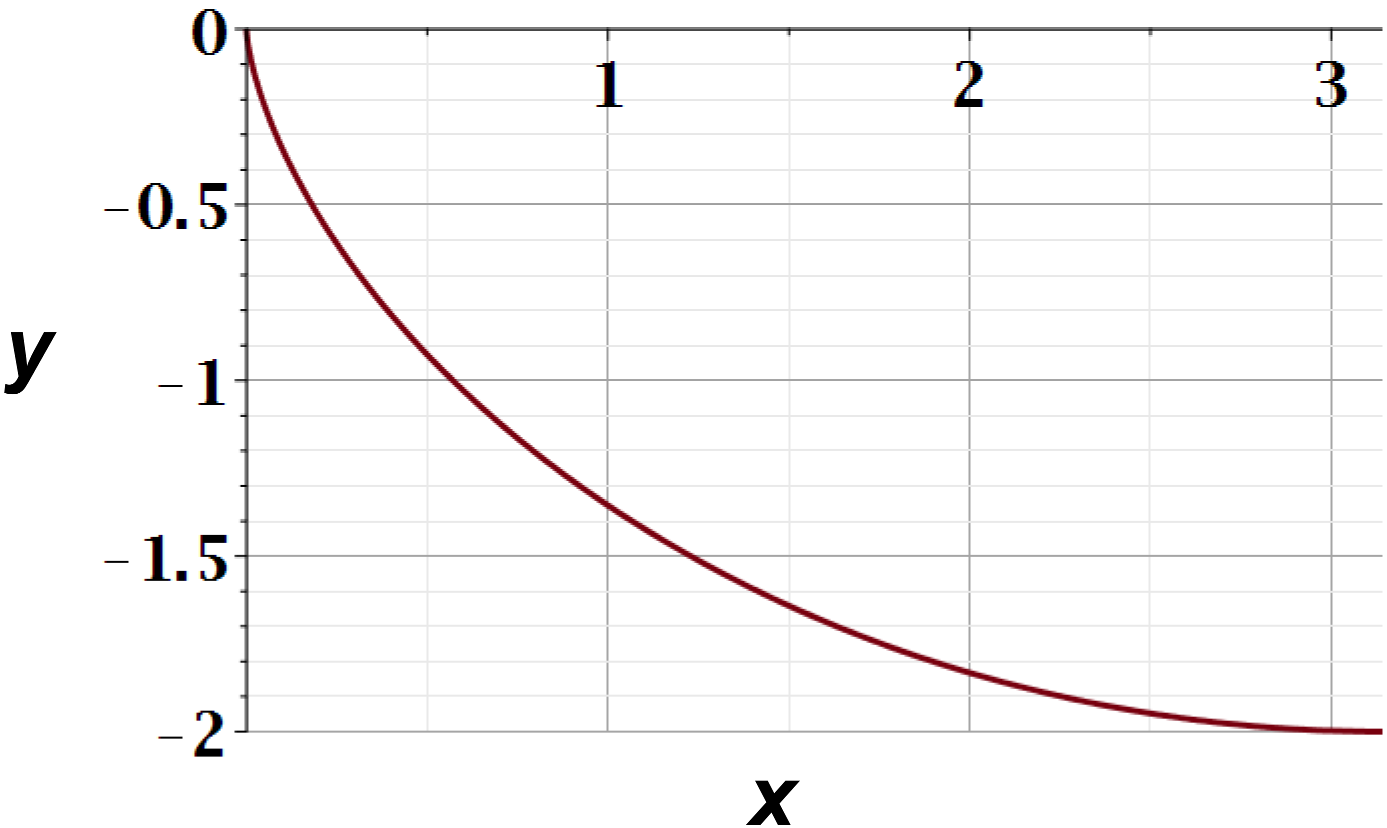
Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$

$$y=a(\cos\theta-1)$$

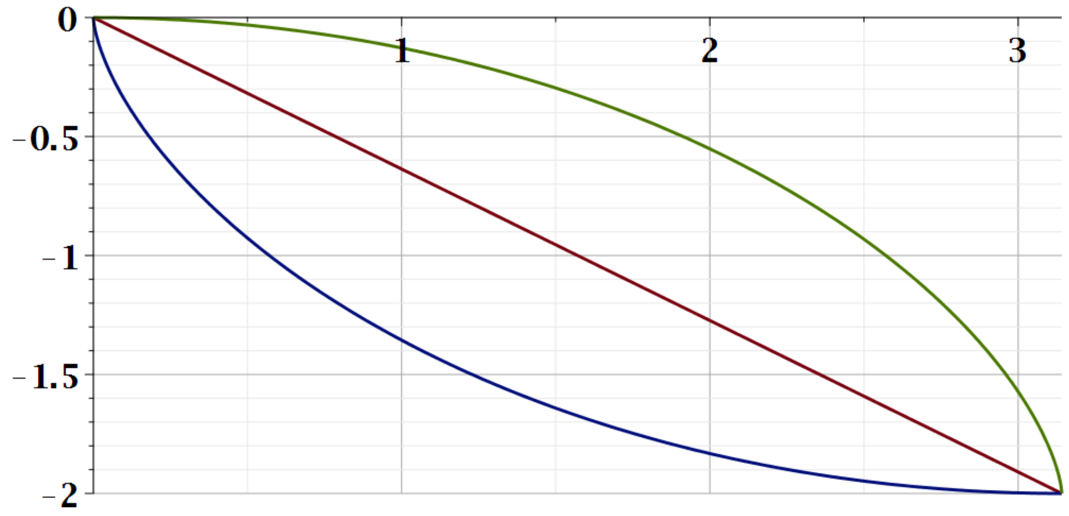
Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\,dx$$



T=infinite
T=5.2668
T=4.4429

$$\text{(units of } \frac{1}{\sqrt{(2g)}} \text{)}$$

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \qquad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Euler-Lagrange equation:

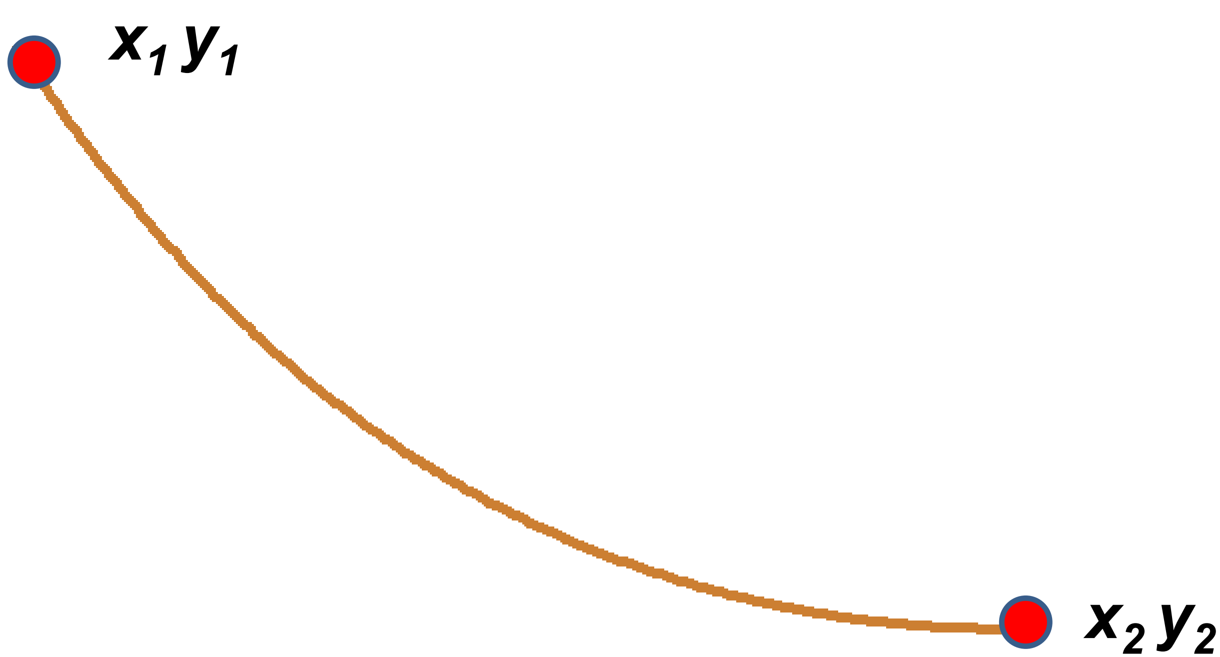
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points



Example from internet --

Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

Your question -- what does W=E+λL stand for?

Comment -- W does not have an obvious physical interpretation

But $\delta W = 0 = \delta E + \lambda \delta L$ for fixed λ is a very clever mathematical trick to help solve the minimization and constraint at the same time.

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$(\rho g y+\lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$(\rho g y+\lambda)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants : K, a, λ

Constraints : $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{ y(x), \frac{dy}{dx} \right\}, x \right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y} \right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial \left(dy/dx \right)} \right) \right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial \left(dy/dx \right)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

Application to particle dynamics

$$x \rightarrow t \qquad \text{(time)}$$

$$y \rightarrow q \qquad \text{(generalized coordinate)}$$

$$f \rightarrow L \qquad \text{(Lagrangian)}$$

$$I \rightarrow A \text{ or } S \qquad \text{(action)}$$

$$\text{Denote:} \quad \dot{q} \equiv \frac{dq}{dt}$$

$$A = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$A = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt \equiv \int_{t_1}^{t_2} L\left(\left\{y, \frac{dy}{dt}\right\}; t\right) dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g*.

Newton's formulation: $m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2} g t^2$

Lagrangian for this case:

$$L = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 - mgy$$

http://www-history.mcs.st-and.ac.uk/Biographies/Hamilton.html

Sir William Rowan Hamilton

Wednesday, September 11th, 2013

Sitemap

- Home
- Biography
- Mathematical studies
- Optics and Dynamics
- Quaternions
- Quotations
- Hamilton Key Dates
- Hamilton Links
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- Math News

Tribute to Sir William Hamilton

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

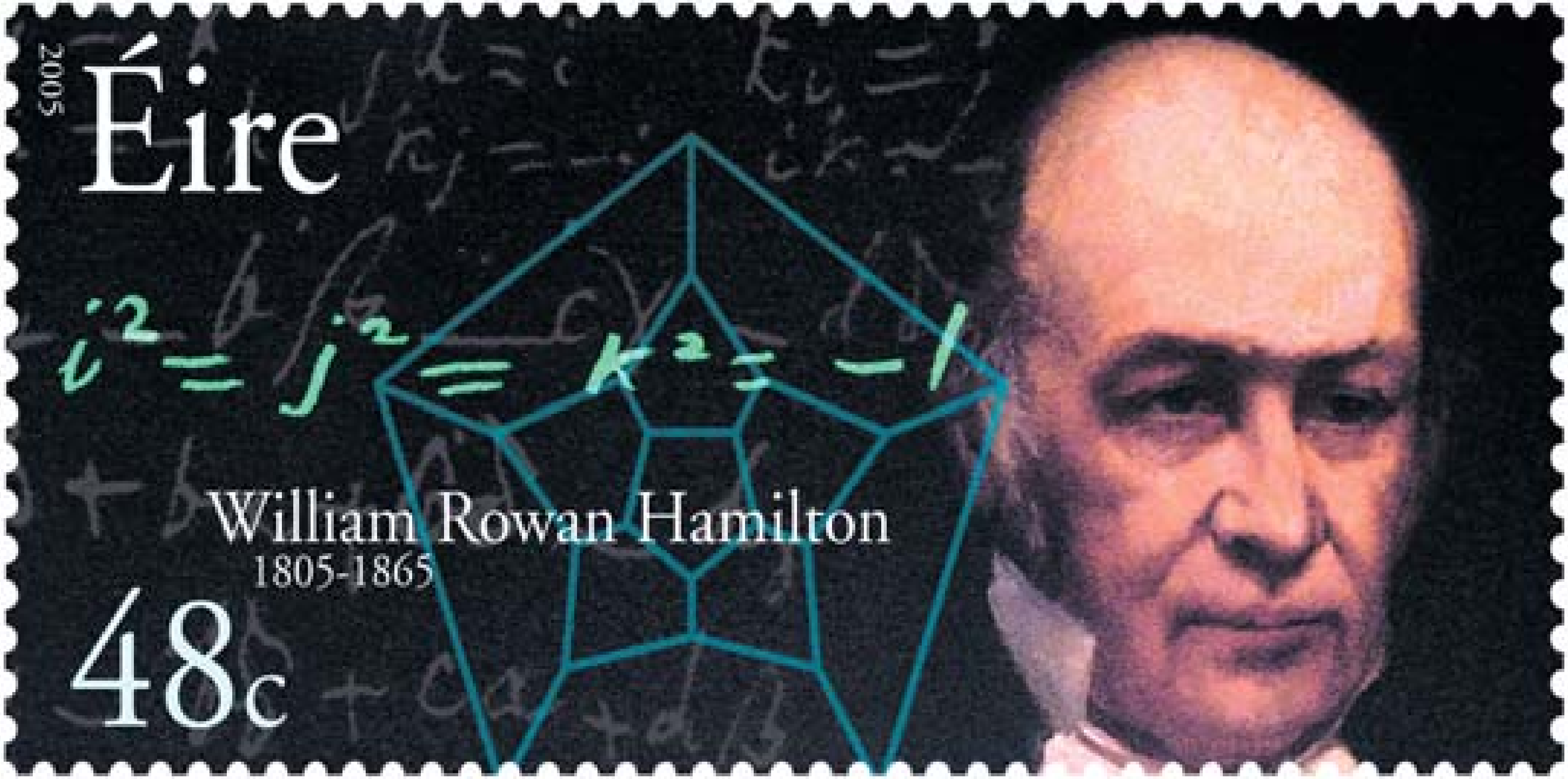
William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

The Discovery of Quaternions is probably is his most familiar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

Thanks for visiting this site! Please enjoy your stay while browsing through the pages.



https://irishpostalheritagegpo.wordpress.com/2017/06/08/william-rowan-hamilton-irish-mathematician-and-scientist/

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy$$

Hamilton's principle states:

$$S\equiv\int_{t_i}^{t_f}L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Check:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0, \; y_i = h \equiv \frac{1}{2} g T^2; \quad t_f = T, \; y_f = 0$

Trial trajectories: $y_1(t) = \frac{1}{2} g T^2 \left(1 - t/T\right) = h - \frac{1}{2} g T t$

$$y_2(t) = \frac{1}{2} g T^2 \left(1 - t^2/T^2\right) = h - \frac{1}{2} g t^2$$

$$y_3(t) = \frac{1}{2} g T^2 \left(1 - t^3/T^3\right) = h - \frac{1}{2} g t^3 / T$$

Maple says:

$$S_1 = -0.125 m g^2 T^3$$

$$S_2 = -0.167 m g^2 T^3$$

$$S_3 = -0.150 m g^2 T^3$$