# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM  MWF Online or (occasionally) in Olin 103

## Plan for Lecture 8 – Chap. 3 F & W

### Calculus of variation

1. Brachistochrone problem
2. Calculus of variation with constraints
3. Application to classical mechanics



In this lecture, we will continue to develop  notions of the calculations of variation and to start to show how they may be applied to classical mechanics.

## Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

| | Date | F&W Reading | Topic | Assignment | Due |
|---|---|---|---|---|---|
| 1 | Wed, 8/26/2020 | Chap. 1 | Introduction | #1 | 8/31/2020 |
| 2 | Fri, 8/28/2020 | Chap. 1 | Scattering theory | #2 | 9/02/2020 |
| 3 | Mon, 8/31/2020 | Chap. 1 | Scattering theory | #3 | 9/04/2020 |
| 4 | Wed, 9/02/2020 | Chap. 1 | Scattering theory | | |
| 5 | Fri, 9/04/2020 | Chap. 1 | Scattering theory | #4 | 9/09/2020 |
| 6 | Mon, 9/07/2020 | Chap. 2 | Non-inertial coordinate systems | | |
| 7 | Wed, 9/09/2020 | Chap. 3 | Calculus of Variation | #5 | 9/11/2020 |
| 8 | Fri, 9/11/2020 | Chap. 3 | Calculus of Variation | #6 | 9/14/2020 |



There is one homework problem for this lecture.

# PHY 711 – Assignment #6

September 7, 2020

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Evaluate
   $$\frac{\partial}{\partial q}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q}.$$

   (b) Evaluate
   $$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}}.$$

   (c) Evaluate
   $$\frac{df}{dt}.$$

   (d) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t^2, \quad \text{where} \quad q(t) = \mathrm{e}^{-t/\tau}.$$

   Here $\tau$ is a constant. Evaluate $df/dt$ using the expression you just derived. Now find the expression for $f$ as an explicit function of $t$ ($f(t)$) and take its time derivative directly to check your previous results.



It might be useful to evaluate part (c) first.

Review: for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

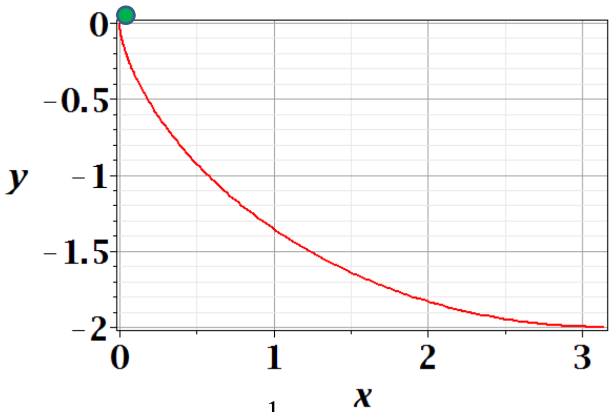
Alternate Euler-Lagrange equation



Summary of the equations we worked out last time.

**Brachistochrone problem:** (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html




A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$ ?

$$E = \frac{1}{2}mv^2 + mgy$$

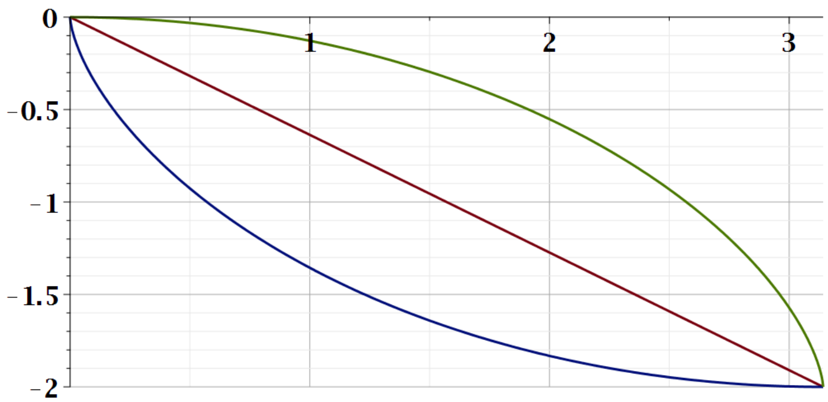
With the choice of initial conditions,

$$E = 0$$



This is the famous problem.

Vote for your favorite path




Which gives the shortest time?

a. Green
b. Red
c. Blue



What curve will win the race?

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\,dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$



Some details of the Euler-Lagrange equations. The green equations look harder.

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

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Calling the integration 2a is very convenient.

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

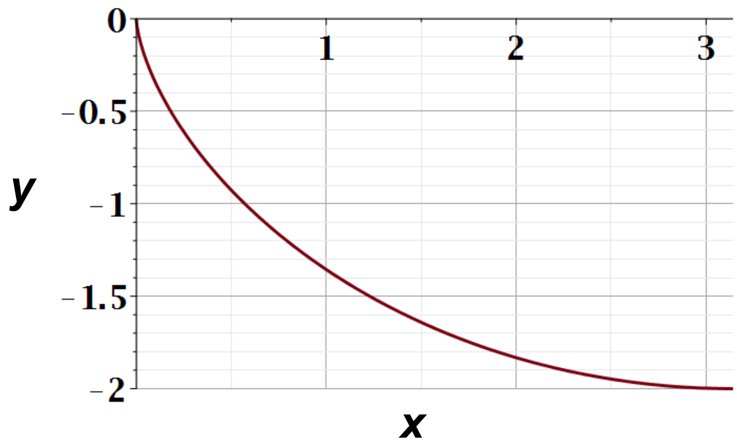
$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$



Very clever mathematics.

Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])

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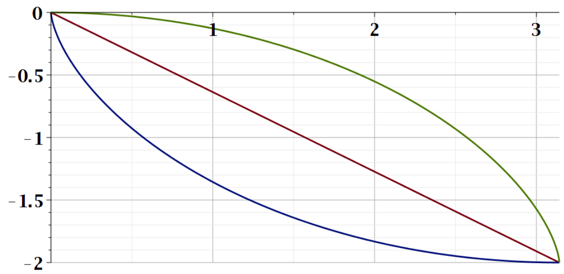
Visualization of the result.

## Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\, dx$$




T=infinite
T=5.2668
T=4.4429

(units of $\sqrt{(2g)}$ )

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How did you do with your bet?

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$



Summary of equations to use.

## Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

## Alternate Euler-Lagrange equation:

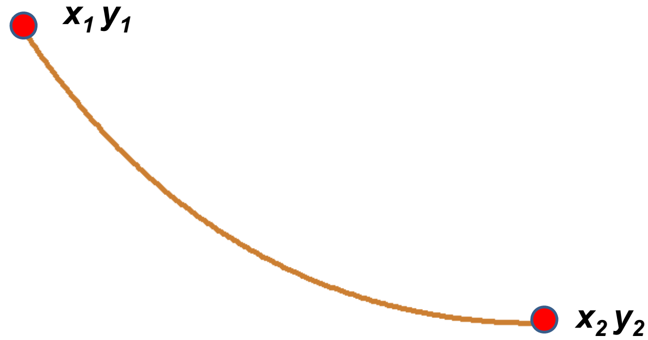
$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$



It is a good idea to remember these equations.

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density $\rho$ hanging between two points

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Another example needing extra information.

Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

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How to minimize with a constraint.

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = K$$



Applying the equations.

$$(\rho g y+\lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$(\rho g y+\lambda)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants : $K, a, \lambda$

Constraints : $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = L$$



The solutions in (almost) convenient form.

## Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$



Summary again.

Application to particle dynamics

$$x \rightarrow t \quad \text{(time)}$$

$$y \rightarrow q \quad \text{(generalized coordinate)}$$

$$f \rightarrow L \quad \text{(Lagrangian)}$$

$$I \rightarrow A \text{ or } S \quad \text{(action)}$$

Denote: $\dot{q} \equiv \dfrac{dq}{dt}$

$$A = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$

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We will now start to apply this mathematics to the physics of motion. Here we map the variables that will apply. A is called “action”. L is called “Lagrangian”.

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$A = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt \equiv \int_{t_1}^{t_2} L\left(\left\{y,\frac{dy}{dt}\right\};t\right)dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g.*

Newton's formulation: $m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2}gt^2$

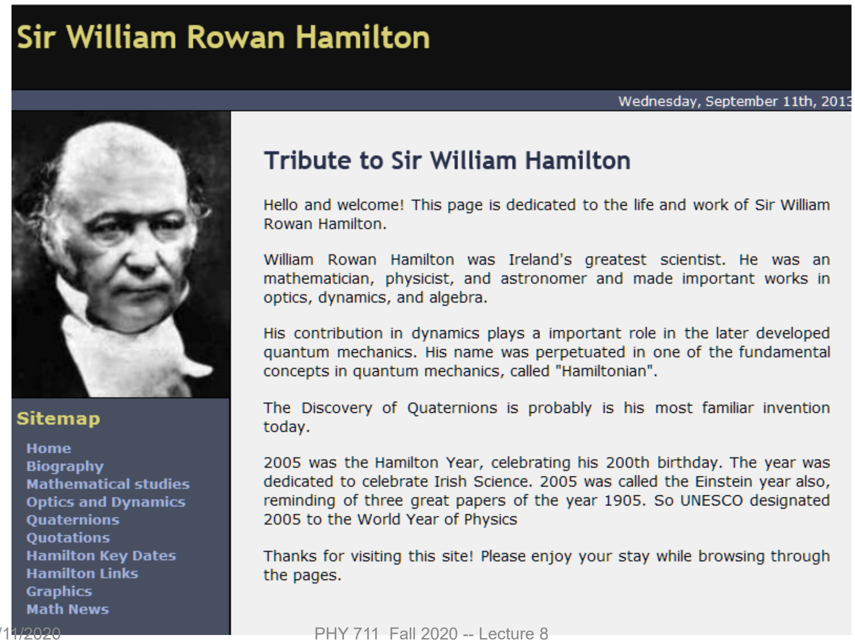
Lagrangian for this case:

$$L = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$



Here we will show how Newton's laws can be written in terms of the Lagrangian formalism.

http://www-history.mcs.st-and.ac.uk/Biographies/Hamilton.html



Sir William Rowan Hamilton

Wednesday, September 11th, 2013

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## Tribute to Sir William Hamilton

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

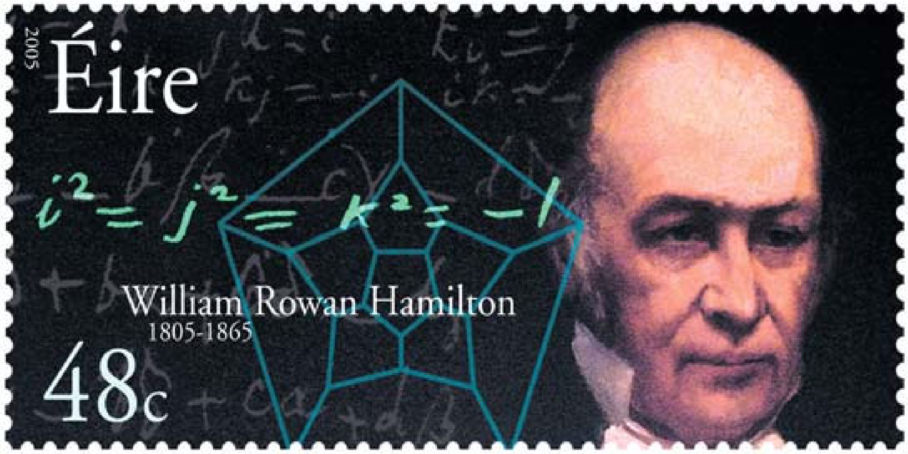
The Discovery of Quaternions is probably is his most familiar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

Thanks for visiting this site! Please enjoy your stay while browsing through the pages.



In addition to Euler and Lagrange, we need to thank Hamilton as well.




https://irishpostalheritagegpo.wordpress.com/2017/06/08/william-rowan-hamilton-irish-mathematician-and-scientist/

Now consider the Lagrangian defined to be:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt \quad \text{is minimized for physical } y(t):$$



First we will show that it works with these relationships and then we will justify how this might work.

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left( \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

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Action is sometimes A and sometimes S.

Check:

$$S \equiv \int_{t_i}^{t_f} \left( \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0,\ \ y_i = h \equiv \frac{1}{2} g T^2; \quad t_f = T,\ \ y_f = 0$

Trial trajectories: $y_1(t) = \frac{1}{2} g T^2 \left(1 - t/T\right) = h - \frac{1}{2} g T t$

$$y_2(t) = \frac{1}{2} g T^2 \left(1 - t^2/T^2\right) = h - \frac{1}{2} g t^2$$

$$y_3(t) = \frac{1}{2} g T^2 \left(1 - t^3/T^3\right) = h - \frac{1}{2} g t^3/T$$

Maple says:

$$S_1 = -0.125 m g^2 T^3$$

$$S_2 = -0.167 m g^2 T^3$$

$$S_3 = -0.150 m g^2 T^3$$



Checking the minimization.