PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Discussion of Lecture 9 – Chap. 3&6 in F&W

Lagrangian mechanics

1. D'Alembert's principle
2. Lagrange's equation in generalized coordinates
3. Examples, including Lagrangian for electromagnetic interactions.

Schedule for weekly one-on-one meetings

Nick – 11 AM Monday (ED/ST)
Tim – 9 AM Tuesday
Bamidele – 7 PM Tuesday
Zhi– 9 PM Tuesday
Jeanette – 11 AM Friday
Derek – 12 PM Friday

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Constants of the motion	#8	9/21/2020

PHY 711 – Assignment #7

September 14, 2020

1. Consider a Lagrangian describing the motion of a particle of mass m and charge q given by

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c}B\dot{y}x.$$

Here c denotes the speed of light and B represents the magnitude of a constant magnetic field along the z-axis. [Note that we are using cgs Gaussian units for electromagnetic interactions following your textbook (see section 33 of Chapter 6).] Determine the Euler-Lagrange equations of motion for the particle and discuss how the motion compares with the similar example discussed in class.

Your questions –

From Tim –

1. What is the meaning of sigma in the slides? Is it like the same as i and so is an indice. Does it have any special meaning in terms of Langrangian mechanics?

From Nick –

1. Slide 7: Can you go over the differential (dx) term and its pieces? What exactly does the differential δq mean?
2. Slide 8: is the expansion a reorganization from the product rule?
3. By about slide 11, I started getting tripped up with some of the algebra/calculus. Hopefully, I can clarify tomorrow.

From Derek –

1. I wasn't able to finish assignment #6 because I'm still confused on how we're supposed to evaluate parts a), b), and c). If you could discuss the assignment more, that would be helpful.

Your questions – continued

From Gao –

1. How many kinds of forces are velocity– dependent, except for Lorentz forces?

From Jeanette –

1. slide #9 - I don't follow the proof for the first claim.
2. slide #14 - where did the equation for L come from? Is the equation given or the situation given and L determined from there?
3. slide #18 - is the switch back to cartesian just convenient or is that necessary for Lorentz forces?

More details about HW 6

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Evaluate
   $$\frac{\partial}{\partial q}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q}.$$

   (b) Evaluate
   $$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}}.$$

   (c) Evaluate
   $$\frac{df}{dt}.$$

   (d) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t^2, \quad \text{where} \quad q(t) = \mathrm{e}^{-t/\tau}.$$

   Here τ is a constant. Evaluate df/dt using the expression you just derived. Now find the expression for f as an explicit function of t ($f(t)$) and take its time derivative directly to check your previous results.

More details about HW 6
From previous lecture:

Suppose that we have a generalized coordinate $q(t)$ that varies with time t and a function that has the dependences $f\left(q(t),\frac{dq}{dt}(t);t\right)$. As an example, suppose that $q(t)=e^{-t/\tau}$ and $f\left(q(t),\dot{q}(t);t\right)=q\dot{q}^2t^2$ Evaluate $\frac{df}{dt}$ in two different ways for (d) and (c). You should get the same answer. For parts (a) and (b) you are asked to take two derivatives in different orders. The results may be surprising.

Further comment --

The point of this problem is to fully understand total and partial derivatives

For $W(q(t),\dot{q}(t),t)$, $\frac{dW}{dt}=\frac{\partial W}{\partial q}\dot{q}+\frac{\partial W}{\partial \dot{q}}\ddot{q}+\frac{\partial W}{\partial t}$

Note that this pattern can be applied to $W \rightarrow \frac{dW}{dt}$

Also note that $\frac{\partial^2 W}{\partial q \partial \dot{q}}=\frac{\partial^2 W}{\partial \dot{q} \partial q}$

On the topic of today's lecture and adapting calculus of variation tools to the analysis of particle motion

Why would we want to reformulate Newton's laws anyway?

a. Because we can and it makes us feel good.
b. Because we can make the analysis more complicated.
c. Because we can make the analysis less complicated.
d. All of the above.
e. Other.

Jean d'Alembert 1717-1783
French mathematician and philosopher

"Deriving" Lagrangian mechanics from Newton's laws.

The Lagrangian function is:

$$L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right)\equiv T-U \quad q_i(t) \text{ are generalized coordinates}$$

Hamilton's principle states:

$$S\equiv\int_{t_i}^{t_f} L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right)dt \quad \text{is minimized for physical } q_i(t):$$

Digression -- notion of generalized coordinates

Referenced to cartesian coordinates: $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$

Cylindrical coordinates

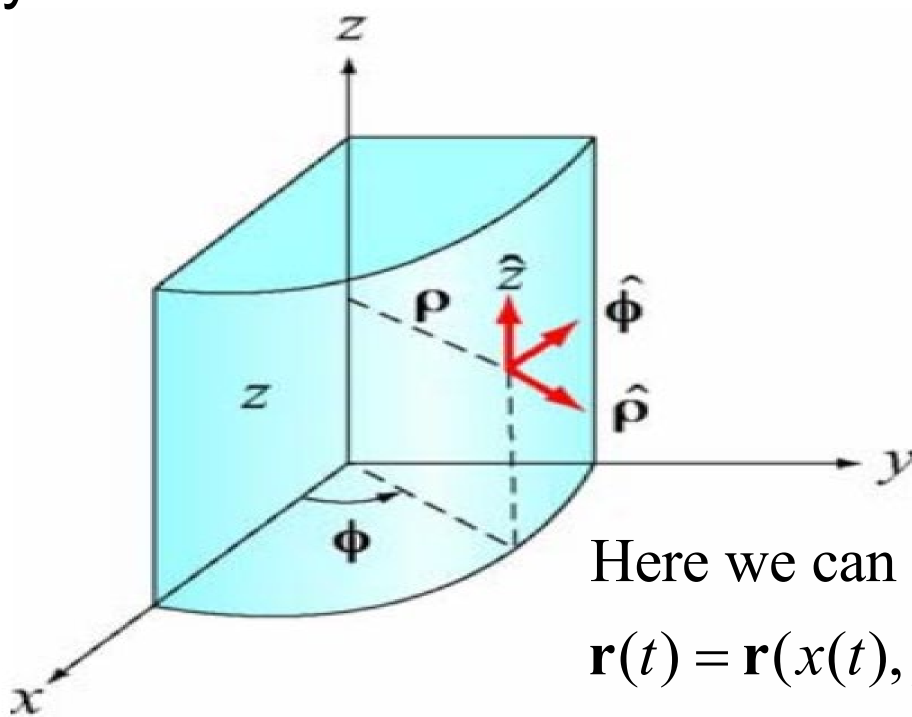


Figure B.2.4 Cylindrical coordinates

$$x = \rho\cos\phi \equiv x(\rho,\phi)$$

$$y = \rho\sin\phi \equiv y(\rho,\phi)$$

$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \arctan(y/x)$$

$$z = z$$

Here we can write

$$\mathbf{r}(t) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(\rho(t), \phi(t), z(t))$$

(Figure taken from 8.02 handout from MIT.)

Spherical coordinates

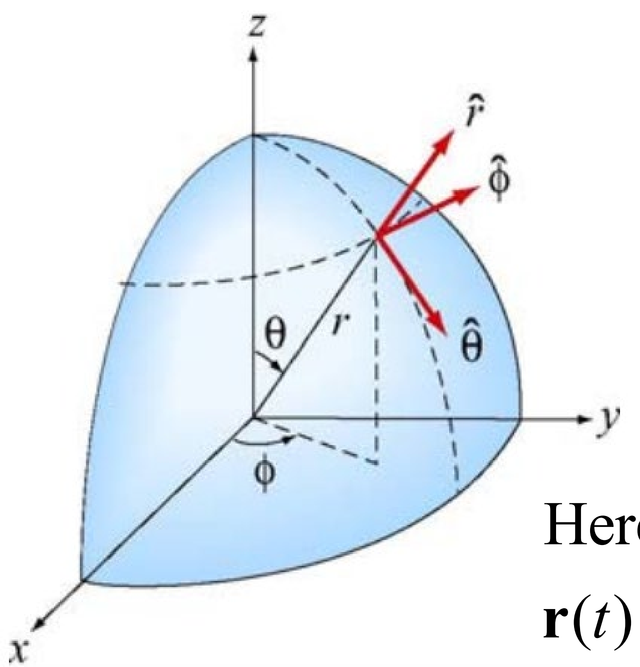


Figure B.3.1 Spherical coordinates

$$x = r\sin\theta\cos\phi \equiv x(r,\theta,\phi)$$

$$y = r\sin\theta\sin\phi \equiv y(r,\theta,\phi)$$

$$z = r\cos\theta \equiv z(r,\theta,\phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

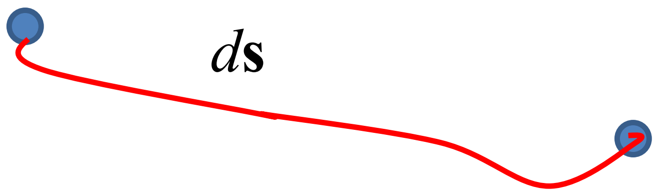
$$\phi = \arctan(y / x)$$

Here we can write

$$\mathbf{r}(t) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(r(t), \theta(t), \phi(t))$$

(Figure taken from 8.02 handout from MIT.)

D'Alembert's principle:



Note that: $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

Newton's laws:

$$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow \left(\mathbf{F}\text{-}m\mathbf{a}\right)\cdot d\mathbf{s} = 0$$

$$\mathbf{F}\cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force: $F_i = -\dfrac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

Generalized coordinates:

$$q_\sigma\left(\{x_i\}\right) \leftrightarrow x_i(\{q_\sigma\})$$

Note that

$q_\sigma(t)$ can be $x(t), \theta(t)$....

$$dx \equiv dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

Your question –
Can you go over the differential (dx) term and its pieces? What exactly does the differential δq mean?

Comment on notation -- $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

For convenience let $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1, \quad \hat{\mathbf{y}} = \hat{\mathbf{x}}_2, \quad \hat{\mathbf{z}} = \hat{\mathbf{x}}_3$

Then $\mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} F_i dx_i$

But now we want to change coordinates $q_\sigma\left(\{x_i\}\right) \leftrightarrow x_i(\{q_\sigma\})$

$$dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \qquad \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} F_i dx_i = \sum_\sigma \sum_{i=1}^{3} F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

Summary up to now --

$$\mathbf{F}\cdot d\mathbf{s}=\sum_{\sigma}\sum_{i}F_i\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force: $F_i=-\dfrac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s}=-\sum_{\sigma}\sum_{i}\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma=-\sum_{\sigma}\frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

Newton's laws:

$\mathbf{F} - m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0$

Generalized coordinates:

$q_\sigma(\{x_i\})$

$x \Leftrightarrow x_1$

$y \Leftrightarrow x_2$

$z \Leftrightarrow x_3$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt} \left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m\dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

Claim: $\displaystyle \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\displaystyle \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt} \left(\frac{\partial \left(\frac{1}{2} m\dot{x}_i^2\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial \left(\frac{1}{2} m\dot{x}_i^2\right)}{\partial q_\sigma} \right) \delta q_\sigma$$

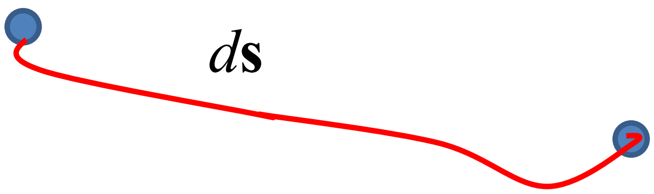
Some details

$$\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} = \frac{d\dot{x}_i}{dt}\frac{\partial x_i}{\partial q_\sigma} = \frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - \dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}$$

You may be wondering why we need to introduce "generalized" coordinates when cartesian coordinates are an example. What the generalized coordinates allow us to show is that

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

where $T \equiv \sum_i \frac{1}{2} m\dot{x}_i^{\,2}$ (kinetic energy)



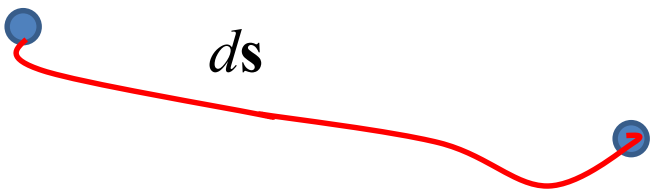
$$x_i = x_i\left(\{q_\sigma(t)\}, t\right)$$

Claim: $\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$

Details: $\dot{x}_i = \sum_\sigma \dfrac{\partial x_i}{\partial q_\sigma}\dot{q}_\sigma + \dfrac{\partial x_i}{\partial t}$ Therefore: $\dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \dfrac{\partial x_i}{\partial q_\sigma}$

Claim: $\dfrac{d}{dt}\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial}{\partial q_\sigma}\dfrac{dx_i}{dt} \equiv \dfrac{\partial \dot{x}_i}{\partial q_\sigma}$

$$\sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_{\sigma'}\partial q_\sigma}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial t \partial q_\sigma} \qquad \sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_\sigma \partial q_{\sigma'}}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial q_\sigma \partial t}$$



Generalized coordinates:

$$q_\sigma\left(\{x_i\}\right)$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2} m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2} m\dot{x}_i^{\,2}\right)}{\partial q_\sigma} \right) \delta q_\sigma$$

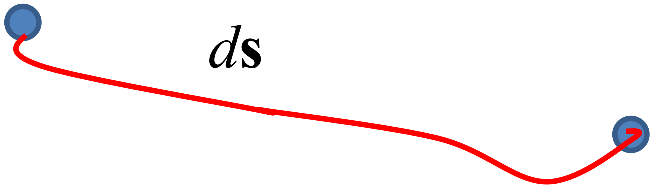
$$\text{Define -- kinetic energy:} \quad T \equiv \sum_i \tfrac{1}{2} m\dot{x}_i^{\,2}$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

Recall:

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$\left(\mathbf{F} - m\mathbf{a}\right)\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$



Generalized coordinates :

$q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

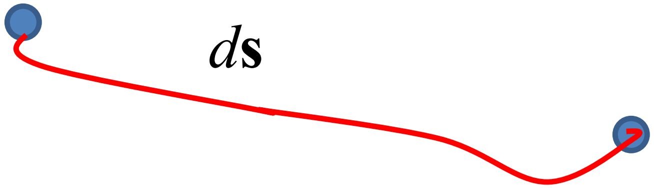
$$= \quad -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$= \quad -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if

$$\frac{\partial U}{\partial \dot{q}_\sigma} = 0$$



Generalized coordinates :

$q_\sigma(\{x_i\})$

$$\text{Define -- Lagrangian}: \quad L \equiv T - U$$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$(\mathbf{F}\text{-}m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$\Rightarrow \text{Minimization integral}: \quad S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$$

➔Hamilton's principle from the "backwards" application of the Euler-Lagrange equations to

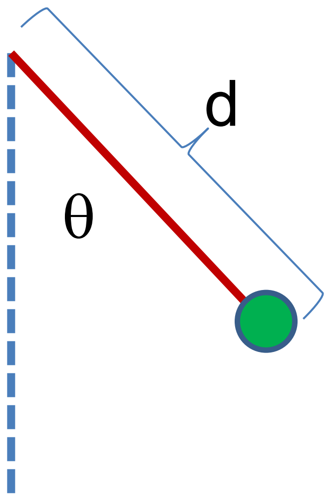
$$\text{Define -- Lagrangian}: \quad L \equiv T - U$$

$$L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right)$$

Euler – Lagrange equations: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta,\dot{\theta}) = \tfrac{1}{2}md^2\dot{\theta}^2 - mg(d - d\cos\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \Rightarrow \frac{d}{dt}md^2\dot{\theta} + mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{d}\sin\theta$$

Another example: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1\dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\right) = 0$$

Example – simple harmonic oscillator

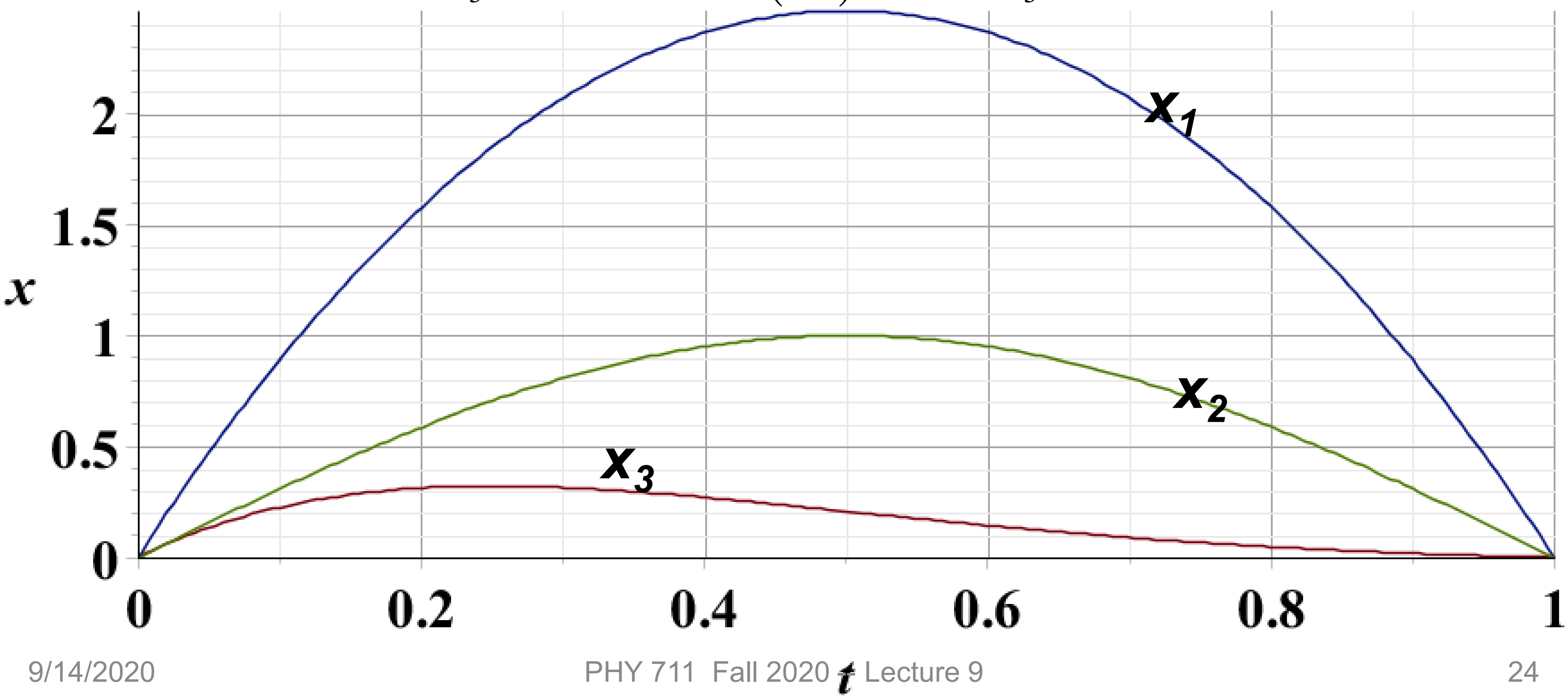
$T = \frac{1}{2} m\dot{x}^2$ $\qquad$ $U = \frac{1}{2} m\omega^2 x^2$

Assume $x(0) = 0$ and $x(\frac{\pi}{\omega}) = 0$ $\qquad$ $S = \frac{1}{2} m \int_0^{\pi/\omega} \left(\dot{x}^2 - \omega^2 x^2\right) dt$

Trial functions $x_1(t) = A\sin(\omega t)$ $\qquad$ $S_1 = 0$

$x_2(t) = A\omega t \cdot (\pi - \omega t)$ $\qquad$ $S_2 = 0.067\, A^2 m\omega^2$

$x_3(t) = Ae^{-\omega t}\sin(\omega t)$ $\qquad$ $S_3 = 0.062\, A^2 m\omega^2$



Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \qquad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

$\Rightarrow$ How can we represent velocity-dependent forces?

Why do we need velocity dependent forces?

a. Friction is sometimes represented as a velocity dependent force. (difficult to treat with Lagrangian mechanics.)
b. Lorentz force on a moving charged particle in the presence of a magnetic field.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

x − component: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Note: Here we are using cartesian coordinates for convenience.

x-component: $$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0$$

Apparently: $$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

Answer: $$U = q\Phi\left(\mathbf{r},t\right) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$$

where $\mathbf{E}\left(\mathbf{r},t\right) = -\nabla\Phi\left(\mathbf{r},t\right) - \frac{1}{c}\frac{\partial \mathbf{A}\left(\mathbf{r},t\right)}{\partial t}$ $\quad$ $\mathbf{B}\left(\mathbf{r},t\right) = \nabla\times\mathbf{A}\left(\mathbf{r},t\right)$

Lorentz forces, continued:

$$x-\text{component of Lorentz force: } F_x = q\left(E_x + \tfrac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$$

$$\text{Suppose:} \qquad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

$$\text{Consider:} \quad F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi\left(\mathbf{r}, t\right) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

where $\mathbf{E}\left(\mathbf{r}, t\right) = -\nabla\Phi\left(\mathbf{r}, t\right) - \frac{1}{c}\frac{\partial \mathbf{A}\left(\mathbf{r}, t\right)}{\partial t} \qquad \mathbf{B}\left(\mathbf{r}, t\right) = \nabla \times \mathbf{A}\left(\mathbf{r}, t\right)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi\left(\mathbf{r}, t\right) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Suppose $\mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Example Lorentz force -- continued

Consider formulation with different Gauge: $\mathbf{A}(\mathbf{r}) = -B_0 y\hat{\mathbf{x}}$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{c} B_0 y \right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y}\right) + \frac{q}{c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

Evaluation of equations:

$$m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0 \qquad \dot{x}(t) = V_0 \sin\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0 \qquad \dot{y}(t) = V_0 \cos\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{z} = 0 \qquad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \tfrac{mc}{qB_0}V_0 \cos\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$y(t) = y_0 + \tfrac{mc}{qB_0}V_0 \sin\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$z(t) = z_0 + V_{0z}t$$