PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Online or (occasionally) in Olin 103

Plan for Lecture 9 – Chap. 3&6 in F&W

Lagrangian mechanics

1. D'Alembert's principle
2. Hamilton's principle
3. Lagrange's equation in generalized coordinates

In this lecture we will continue to examine how the calculus of variation can be used to analyze the physical mechanics of particle motion. This material follows your textbook in both Chapter 3 and Chapter 6.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Wed, 8/26/2020	Chap. 1	Introduction	#1	8/31/2020
2	Fri, 8/28/2020	Chap. 1	Scattering theory	#2	9/02/2020
3	Mon, 8/31/2020	Chap. 1	Scattering theory	#3	9/04/2020
4	Wed, 9/02/2020	Chap. 1	Scattering theory		
5	Fri, 9/04/2020	Chap. 1	Scattering theory	#4	9/09/2020
6	Mon, 9/07/2020	Chap. 2	Non-inertial coordinate systems		
7	Wed, 9/09/2020	Chap. 3	Calculus of Variation	#5	9/11/2020
8	Fri, 9/11/2020	Chap. 3	Calculus of Variation	#6	9/14/2020
9	Mon, 9/14/2020	Chap. 3 & 6	Lagrangian Mechanics	#7	9/18/2020
10	Wed, 9/16/2020	Chap. 3 & 6	Constants of the motion	#8	9/21/2020

Here is the updated schedule. Note that HW 7 which will be covered in today's lecture will be due on Friday.

PHY 711 – Assignment #7

September 14, 2020

1. Consider a Lagrangian describing the motion of a particle of mass m and charge q given by

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c}B\dot{y}x.$$

   Here c denotes the speed of light and B represents the magnitude of a constant magnetic field along the z-axis. [Note that we are using cgs Gaussian units for electromagnetic interactions following your textbook (see section 33 of Chapter 6).] Determine the Euler-Lagrange equations of motion for the particle and discuss how the motion compares with the similar example discussed in class.

Homework problem due on Friday.

Jean d'Alembert 1717-1783

French mathematician and philosopher

"Deriving" Lagrangian mechanics from Newton's laws.

The Lagrangian function is:

$$L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right)\equiv T-U \quad q_i(t) \text{ are generalized coordinates}$$

Hamilton's principle states:

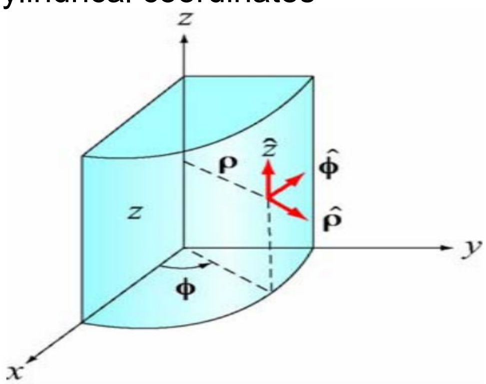
$$S\equiv\int_{t_i}^{t_f} L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right)dt \quad \text{is minimized for physical } y(t):$$

Previously we introduced the Lagrangian function without justification. Here we follow a "derivation" attributed to d'Alembert.

Digression -- notion of generalized coordinates

Referenced to cartesian coordinates: $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$

Cylindrical coordinates



$$x = \rho\cos\phi \equiv x(\rho,\phi)$$
$$y = \rho\sin\phi \equiv y(\rho,\phi)$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \arctan(y/x)$$
$$z = z$$

Figure B.2.4 Cylindrical coordinates

(Figure taken from 8.02 handout from MIT.)

The derivation is based on the notion of "generalized" coordinates which can be Cartesian coordinates or one of the many transformed coordinates, or even more "general" coordinates.

Spherical coordinates

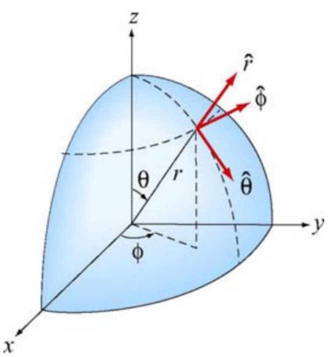


Figure B.3.1 Spherical coordinates

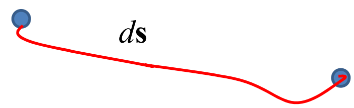
$$x = r\sin\theta\cos\phi \equiv x(r,\theta,\phi)$$
$$y = r\sin\theta\sin\phi \equiv y(r,\theta,\phi)$$
$$z = r\cos\theta \equiv z(r,\theta,\phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$
$$\phi = \arctan(y / x)$$

(Figure taken from 8.02 handout from MIT.)

Another example of transformed coordinates.

D'Alembert's principle:



Generalized coordinates:

$q_\sigma(\{x_i\})$

Note that: $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

$$dx \equiv dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

Newton's laws:

$$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = 0$$

$$\mathbf{F}\cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force: $F_i = -\frac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

Here we start the derivation following D'Alembert's arguments. x_i denotes the cartesian coordinate while q denotes the "generalized" coordinate. In this slide we consider the potential energy terms.

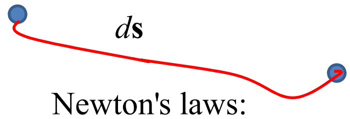
Generalized coordinates:

$q_\sigma\left(\{x_i\}\right)$

$x \Leftrightarrow x_1$

$y \Leftrightarrow x_2$

$z \Leftrightarrow x_3$



Newton's laws:

$$\mathbf{F} - m\mathbf{a} = 0 \qquad \Rightarrow \left(\mathbf{F} - m\mathbf{a}\right)\cdot d\mathbf{s} = 0$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt}\left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma}\right) - m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma$$

Claim: $\displaystyle \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\displaystyle \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma}\frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma}\right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

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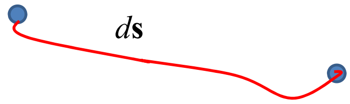
Continuing the derivations we consider the kinetic energy contributions.



$$x_i = x_i\left(\left\{q_\sigma(t)\right\}, t\right)$$

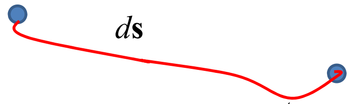
Claim: $\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$

Details: $\dot{x}_i = \sum_\sigma \dfrac{\partial x_i}{\partial q_\sigma}\dot{q}_\sigma + \dfrac{\partial x_i}{\partial t}$ Therefore: $\dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \dfrac{\partial x_i}{\partial q_\sigma}$

Claim: $\dfrac{d}{dt}\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial}{\partial q_\sigma}\dfrac{d x_i}{dt} \equiv \dfrac{\partial \dot{x}_i}{\partial q_\sigma}$

$$\sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_{\sigma'}\partial q_\sigma}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial t \partial q_\sigma} \qquad \sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_\sigma \partial q_{\sigma'}}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial q_\sigma \partial t}$$

Continuing the derivation.



Generalized coordinates:

$q_\sigma(\{x_i\})$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial q_\sigma} \right)\delta q_\sigma$$

Define -- kinetic energy: $T \equiv \sum_i \frac{1}{2}m\dot{x}_i^{\,2}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma$$

Recall:

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$(\mathbf{F} - m\mathbf{a})\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

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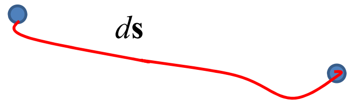
Summary of results from D'Alembert's analysis.



Generalized coordinates: $q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

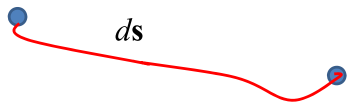
$$= -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$= -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note: This is only true if $\frac{\partial U}{\partial \dot{q}_\sigma} = 0$

Form of derived Lagrangian provided that the potential does not depend on velocity.



Generalized coordinates:
$q_\sigma(\{x_i\})$

Define-- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$(\mathbf{F}\text{-}m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$\Rightarrow \text{Minimization integral:} \quad S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)\, dt$$

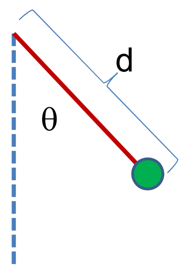
➔Hamilton’s principle

Having shown that the Euler-Lagrangian equations are consistent with Newton’s equations of motion, we can then infer that the integral of the Lagrangian is optimized as is consistent with Hamilton’s principle.

Euler – Lagrange equations : $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L\left(\theta,\dot{\theta}\right) = \tfrac{1}{2} m d^2 \dot{\theta}^2 - mg\left(d - d\cos\theta\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} + mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{d}\sin\theta$$

Example of using the Lagrangian formalism for a simple pendulum.

Another example: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \tfrac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma}\right)^2 - Mgd \cos \beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1 \dot{\alpha} \sin^2 \beta + I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma}\right) \cos \beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1 \dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma}\right)\right) = 0$$

Another example of Lagrangian formalism that we will encounter when we examine rigid body motion.

Example – simple harmonic oscillator

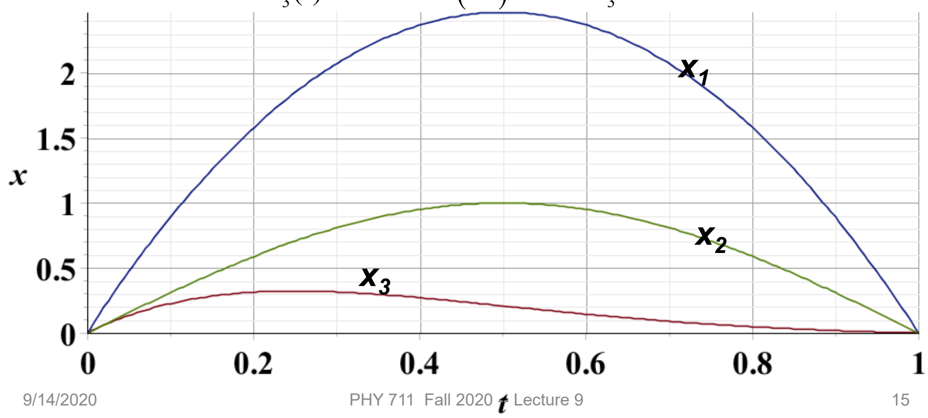
$T = \frac{1}{2} m\dot{x}^2 \qquad U = \frac{1}{2} m\omega^2 x^2$

Assume $x(0) = 0$ and $x(\frac{\pi}{\omega}) = 0 \qquad S = \frac{1}{2} m \int_0^{\pi/\omega} \left(\dot{x}^2 - \omega^2 x^2\right) dt$

Trial functions $x_1(t) = A\sin(\omega t) \qquad S_1 = 0$

$x_2(t) = A\omega t \cdot (\pi - \omega t) \qquad S_2 = 0.067 A^2 m\omega^2$

$x_3(t) = Ae^{-\omega t}\sin(\omega t) \qquad S_3 = 0.062 A^2 m\omega^2$



Harmonic oscillator example. Here we again demonstrate the physical trajectory has the smallest “action”.

Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Recipe for Lagrangian mechanics.

Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

$\Rightarrow$ How can we represent velocity-dependent forces?

Important restriction.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

$x-$component: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L(x,y,z,\dot{x},\dot{y},\dot{z},t) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

x-component: $$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0$$

Apparently: $$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

Answer: $$U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

where $$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

While Lagrangian mechanics cannot treat all velocity dependent forces, it is possible to extend the analysis for the case of the Lorentz force. This material is treated in Chapter 6, Section 33 of your textbook. We are following the textbook's units of cgs Gaussian units.

Lorentz forces, continued:

$$x-\text{component of Lorentz force}: \; F_x = q\left(E_x + \tfrac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$$

$$\text{Suppose}: \quad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

$$\text{Consider}: \; F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Very clever mathematicians figured out how to incorporate Lorentz into the Lagrangian formalism. Here we are assuming their result and showing that it is consistent.

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

More derivations.

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \dfrac{1}{c}\dfrac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Summary of results.

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Suppose $\mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Example for a magnetic field in the z direction.

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c} B_0 y\right) - \frac{q}{2c} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c} B_0 x\right) + \frac{q}{2c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Finding the Euler-Lagrange equations.

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Summary from previous slides.

Example Lorentz force -- continued

Consider formulation with different Gauge: $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{c} B_0 y \right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y}\right) + \frac{q}{c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

This is the same magnetic field, but an equivalent vector potential.

Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0 \qquad \dot{x}(t) = V_0 \sin\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0 \qquad \dot{y}(t) = V_0 \cos\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{z} = 0 \qquad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \tfrac{mc}{qB_0}V_0 \cos\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$y(t) = y_0 + \tfrac{mc}{qB_0}V_0 \sin\left(\tfrac{qB_0}{mc}t + \phi\right)$$

$$z(t) = z_0 + V_{0z}t$$

We get the same motion for this case.