# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103

# Discussion of Lecture 10 – Chap. 3&6 in F&W

## Lagrangian mechanics

1. Lagrange's equations in the presence of velocity dependent potentials – such as electromagnetic interactions.

2. Effects of constraints

# PHY 711 Classical Mechanics and Mathematical Methods

| MWF 10 AM-10:50 AM | OPL 103 | http://www.wfu.edu/~natalie/f21phy711/ |
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## Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

| | Date | F&W Reading | Topic | Assignment | Due |
|---|---|---|---|---|---|
| 1 | Mon, 8/23/2021 | Chap. 1 | Introduction | #1 | 8/27/2021 |
| 2 | Wed, 8/25/2021 | Chap. 1 | Scattering theory | #2 | 8/30/2021 |
| 3 | Fri, 8/27/2021 | Chap. 1 | Scattering theory | | |
| 4 | Mon, 8/30/2021 | Chap. 1 | Scattering theory | #3 | 9/01/2021 |
| 5 | Wed, 9/01/2021 | Chap. 1 | Summary of scattering theory | #4 | 9/03/2021 |
| 6 | Fri, 9/03/2021 | Chap. 2 | Non-inertial coordinate systems | #5 | 9/06/2021 |
| 7 | Mon, 9/06/2021 | Chap. 3 | Calculus of Variation | #6 | 9/10/2021 |
| 8 | Wed, 9/08/2021 | Chap. 3 | Calculus of Variation | | |
| 9 | Fri, 9/10/2021 | Chap. 3 & 6 | Lagrangian Mechanics | #7 | 9/13/2021 |
| 10 | Mon, 9/13/2021 | Chap. 3 & 6 | Lagrangian Mechanics | #8 | 9/17/2021 |

# PHY 711 – Assignment #8

The material for this exercise is covered in the lecture notes and in Chapters 3 and 6 of Fetter and Walecka.

1. A particle of mass $m$ and charge $q$ is subjected to a vector potential $\mathbf{A}(\mathbf{r}, t) = -(E_0 ct + B_0 x)\hat{\mathbf{z}}$. (Note that we are using the cgs Gaussian units of your text book.) Here $E_0$ denotes a constant electric field amplitude and $B_0$ denotes a constant magnetic field amplitude. The initial particle position is $\mathbf{r}(0) = 0$ and the initial particle velocity is $\dot{\mathbf{r}}(0) = 0$.

   (a) Determine the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ which describes the particle's motion.

   (b) Write the Euler-Lagrange equations for this system.

   (c) Find and evaluate the constants of motion for this system.

   (d) Find the particle trajectories $x(t)$, $y(t)$, $z(t)$ by solving the equations and imposing the given initial conditions.
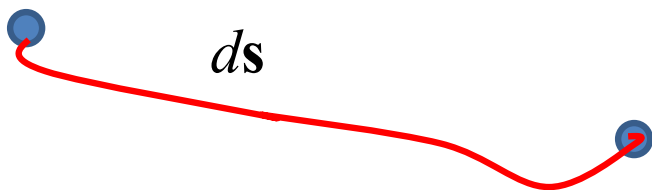
# Your questions –
## From Owen –
1. I have heard a lot about gauge theory but lack a knowledge in the underlying math and formalism to really understand it. How does the Lagrangian formalism we are thinking about now build up to the more advanced topics that are related to gauge theory?

## From Can –
1. Why there is a 1/c in lorentz force.

Previously derived form for the Lagrangian --



Generalized coordinates :

$$q_\sigma(\{x_i\})$$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_\sigma \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$
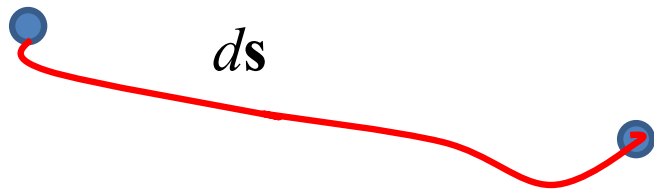
$$= -\sum_\sigma \left( \frac{d}{dt} \frac{\partial (T - U)}{\partial \dot{q}_\sigma} - \frac{\partial (T - U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$= -\sum_\sigma \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if

$$\frac{\partial U}{\partial \dot{q}_\sigma} = 0$$

$d\mathbf{s}$

Generalized coordinates :

$$q_\sigma\left(\{x_i\}\right)$$

Define - - Lagrangian : $\quad L \equiv T - U$

$$L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right)$$

$$\left(\mathbf{F}\text{-}m\mathbf{a}\right) \cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$$\Rightarrow \text{Minimization integral}: \quad S = \int_{t_i}^{t_f} L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right)dt$$

➔ Hamilton's principle from the "backwards" application of the Euler-Lagrange equations to

Define -- Lagrangian: $\quad L \equiv T - U$

$$L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right)$$

Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) \equiv T - U$,

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

are those which minimize the action: $S = \int L(\{q_\sigma\},\{\dot{q}_\sigma\},t)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \qquad \Rightarrow \text{ for each } \sigma: \quad \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Note: in "proof" of Hamilton's principle:

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) \equiv T - U$$

It was necessary to assume that :

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\sigma} \quad \text{does not contribute to the result.}$$

$\Rightarrow$ How can we represent velocity-dependent forces?

Why do we need velocity dependent forces?
  a. Friction is sometimes represented as a velocity dependent force. (difficult to treat with Lagrangian mechanics.)
  b. Lorentz force on a moving charged particle in the presence of a magnetic field.

## Lorentz forces:

For particle of charge $q$ in an electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \tfrac{1}{c}\mathbf{v} \times \mathbf{B}\right)$

$x$ – component: $F_x = q\left(E_x + \tfrac{1}{c}(\mathbf{v} \times \mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$

$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

Note: Here we are using cartesian coordinates for convenience.

$x$-component: $\left(\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x}\right) = 0$

Apparently: $F_x = -\dfrac{\partial U}{\partial x} + \dfrac{d}{dt}\dfrac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi(\mathbf{r}, t) - \dfrac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

where $\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \dfrac{1}{c}\dfrac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$      $\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

# Units for electromagnetic fields and forces

## cgs Gaussian units -- (as used your textbook)

$\mathbf{E}$ and $\mathbf{B}$ fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

## SI units --

$\mathbf{E}$ and $\mathbf{B}$ fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r},t) - q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Lorentz forces, continued:

$x$ − component of Lorentz force : $F_x = q\left(E_x + \frac{1}{c}\left(\mathbf{v} \times \mathbf{B}\right)_x\right)$

Suppose : $\quad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$

Consider : $\quad F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left( \dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left( \frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left( \frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left( \frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left( \frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left( \frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left( \dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left( \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

## Some details on last step:

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

Note that:   $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$        $\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

So that:

$$F_x(\mathbf{r},t) \quad = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi\left(\mathbf{r}, t\right) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

where $\quad \mathbf{E}\left(\mathbf{r}, t\right) = -\nabla\Phi\left(\mathbf{r}, t\right) - \dfrac{1}{c}\dfrac{\partial\mathbf{A}\left(\mathbf{r}, t\right)}{\partial t} \qquad \mathbf{B}\left(\mathbf{r}, t\right) = \nabla \times \mathbf{A}\left(\mathbf{r}, t\right)$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi\left(\mathbf{r}, t\right) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Suppose $\mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c} B_0 y\right) - \frac{q}{2c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c} B_0 x\right) + \frac{q}{2c} B_0 \dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt} m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c} B_0 y\right) - \frac{q}{2c} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c} B_0 x\right) + \frac{q}{2c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained
from direct application of Newton's laws :

$$m\ddot{\mathbf{r}} = \frac{q}{c} \dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \qquad \dot{x}(t) = V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \qquad \dot{y}(t) = V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right)$$

$$m\ddot{z} = 0 \qquad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \frac{mc}{qB_0} V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right)$$

$$y(t) = y_0 + \frac{mc}{qB_0} V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right)$$

$$z(t) = z_0 + V_{0z} t$$

Example Lorentz force -- continued

Consider formulation with different Gauge:   $\mathbf{A}(\mathbf{r}) = -B_0 y\hat{\mathbf{x}}$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{c}B_0 y\right) = 0 \qquad\qquad \Rightarrow m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y}\right) + \frac{q}{c}B_0\dot{x} = 0 \qquad\qquad \Rightarrow m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0$$

$$\frac{d}{dt}m\dot{z} = 0 \qquad\qquad\qquad\qquad \Rightarrow m\ddot{z} = 0$$

Does it surprise you that the same equations of motion are obtained with a different Gauge?

How do these two different forms of **A** correspond to the same **B**?

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

Consider $\quad \mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla f(\mathbf{r},t)$

Note that $\quad \nabla \times \mathbf{A}(\mathbf{r},t) = \nabla \times \mathbf{A}'(\mathbf{r},t)$

In our case, $\mathbf{A}(\mathbf{r},t) = \dfrac{1}{2} B_0 \left( -y\hat{\mathbf{x}} + x\hat{\mathbf{y}} \right)$

$$\mathbf{A}'(\mathbf{r},t) = -B_0 y\hat{\mathbf{x}}$$

What is $f(\mathbf{r},t)$?

Now consider formulation of motion with constraints --
  Comments on generalized coordinates:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates $q_\sigma$ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrangian : $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

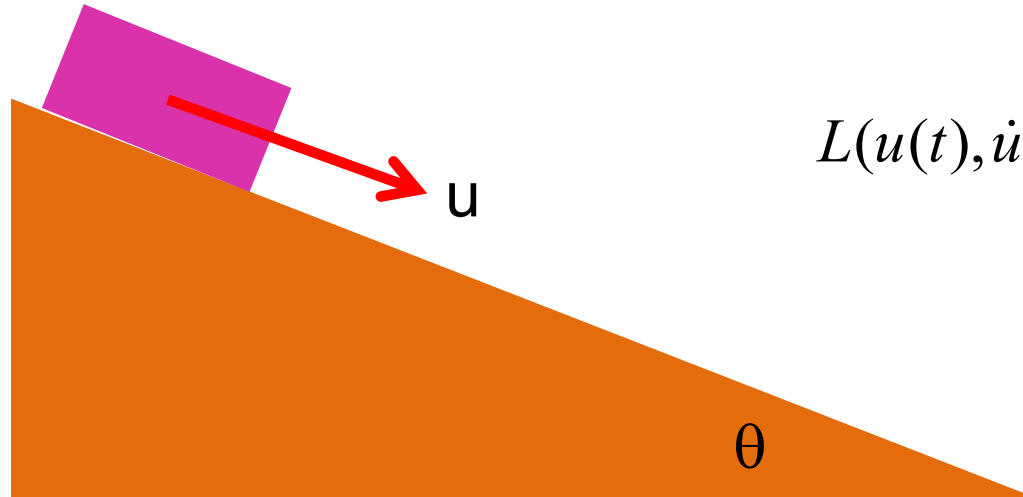Constraints : $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Lagrange multipliers

Modified Euler - Lagrange equations : $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_\sigma} - \dfrac{\partial L}{\partial q_\sigma} + \displaystyle\sum_j \lambda_j \dfrac{\partial f_j}{\partial q_\sigma} = 0$
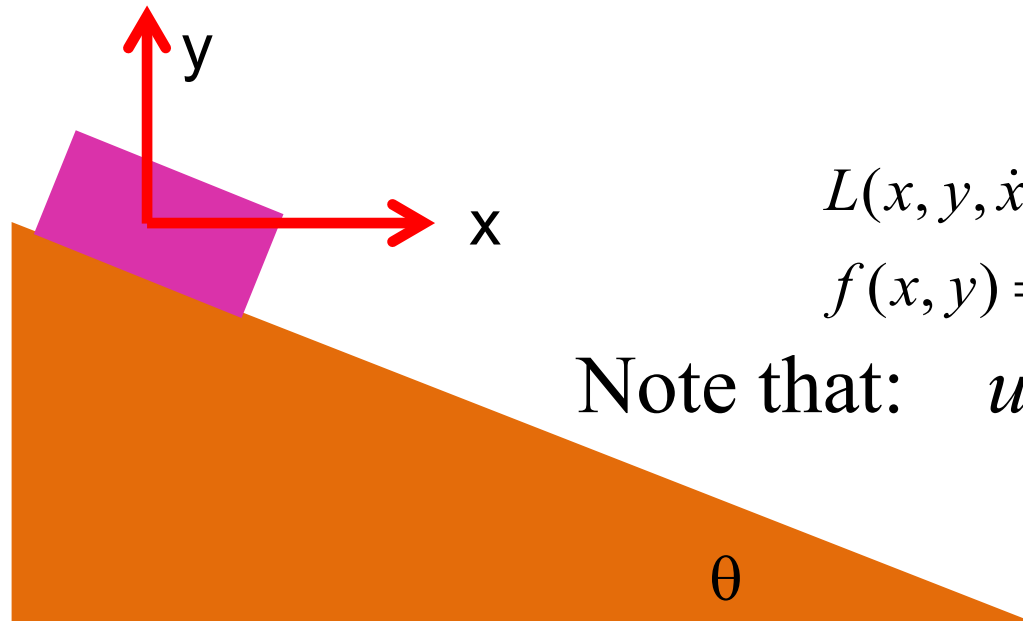
Simple example:



$$L(u(t), \dot{u}(t)) = \tfrac{1}{2} m\dot{u}^2 + mgu\sin\theta$$

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x, y) = \sin\theta\, x + \cos\theta\, y = 0$$

Note that:   $u = x\cos\theta - y\sin\theta$

Case 1:

$$L(u(t), \dot{u}(t)) = \tfrac{1}{2} m \dot{u}^2 + mgu \sin \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg \sin \theta = 0$$

$$\Rightarrow \ddot{u} = g \sin \theta$$

Case 2:

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) - mgy$$

$$f(x, y) = \sin \theta \, x + \cos \theta \, y = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda \sin \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda \cos \theta$$

$$\sin \theta \, \ddot{x} + \cos \theta \, \ddot{y} = 0$$

$$\Rightarrow \lambda = -mg \cos \theta$$

$$\left( \cos \theta \, \ddot{x} - \sin \theta \, \ddot{y} \right) = g \sin \theta$$

Which method would you use to solve the problem?
Case 1
Case 2

Force of constraint; normal to incline

# Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left( \sum_\sigma \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints : $\quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Variations $\delta q_\sigma$ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$
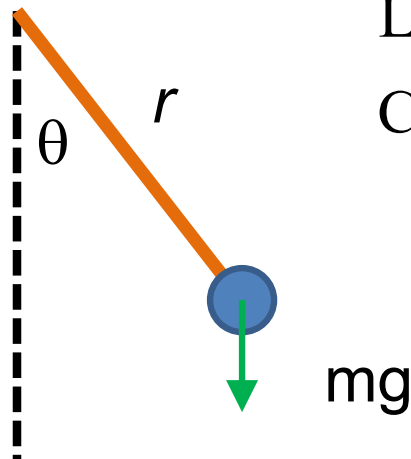
# Euler-Lagrange equations with constraints:

Lagrangian : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints : $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

Modified Euler - Lagrange equations : $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_\sigma} - \dfrac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \dfrac{\partial f_j}{\partial q_\sigma} = 0$

## Example:



Lagrangian : $L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$

Constraints : $f = r - \ell = 0$

## Example continued:

Lagrangian : $L = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + mgr\cos\theta$

Constraints : $f = r - \ell = 0$

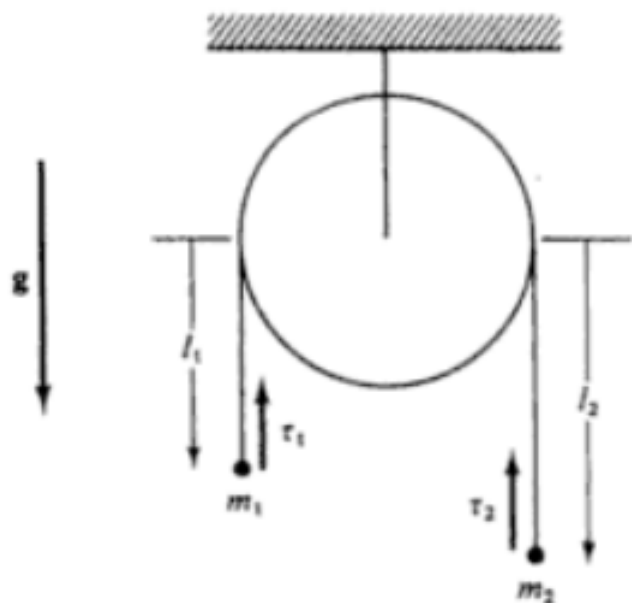$$\frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\frac{d}{dt} mr^2\dot{\theta} + mgr\sin\theta = 0$$

$$\dot{r} = 0 = \ddot{r} \qquad r = \ell$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$\Rightarrow \quad \lambda = m\ell\dot{\theta}^2 + mg\cos\theta$$

Another example:



Figure 19.1  Atwood's machine.

Lagrangian :  $L = \frac{1}{2} m_1 \dot{\ell}_1^2 + \frac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$

Constraints :  $f = \ell_1 + \ell_2 - \ell = 0$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2 m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$