



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Notes on Lecture 11-- Chap. 3 & 6 (F &W)

Details and extensions of Lagrangian mechanics

- 1. Constants of the motion**
- 2. Conserved quantities**
- 3. Legendre transformations**



Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	#8	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion		





Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = (\text{constant})$$



Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \quad \Rightarrow m\dot{x} \equiv p_x \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \quad \Rightarrow m\dot{y} \equiv p_y \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$



Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r)$$

$$\Rightarrow \frac{d}{dt} m r^2 \dot{\phi} = 0 \quad \Rightarrow m r^2 \dot{\phi} \equiv p_{\phi} \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt} m \dot{r} = m r \dot{\phi}^2 - \frac{\partial V}{\partial r} = \frac{p_{\phi}^2}{m r^3} - \frac{\partial V}{\partial r}$$



Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that: $\frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma} \dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \ddot{q}_\sigma + \frac{\partial L}{\partial t}$

$$= \frac{d}{dt} \left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t}$$

Additional constant of the motion:

$$\text{If } \frac{\partial L}{\partial t} = 0;$$

$$\text{then: } \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 1: one - dimensional potential :

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(z) \right) = -E \quad (\text{constant})$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \dot{z}^2 + V(z)$$

Summary from previous slide

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \quad \rightarrow 3 \text{ variable functions}$$

$$E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\dot{z}^2 + V(z) \quad p_x, p_y, E \text{ constant}$$

$\rightarrow 1$ variable function

Why might this be useful?



Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t} = 0$;

then : $\frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) \quad \rightarrow 2 \text{ variable functions}$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) - m\dot{r}^2 - mr^2 \dot{\phi}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r) \right) = -E \quad (\text{constant})$$

For this case, we also have $mr^2 \dot{\phi} \equiv p_{\phi}$

$$\Rightarrow E = \frac{p_{\phi}^2}{2mr^2} + \frac{1}{2} m \dot{r}^2 + V(r) \quad \rightarrow 1 \text{ variable function}$$



Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad m\dot{z} = p_z \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$- \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\Rightarrow E = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{p_z^2}{2m}$$



Other examples

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow m\dot{z} = p_z \quad (\text{constant})$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow m\dot{x} = p_x \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}B_0\dot{x}y$$

$$- \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}B_0\dot{x}y$$

$$\Rightarrow E = \frac{1}{2}m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Your question –

What is the reasoning behind doing a change of variables for the Lagrangian?

Comment –

We are leading up to deriving the Hamiltonian formulation of mechanics which is a useful alternative analysis tool.



Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x, y) \Leftrightarrow x(y, z) ???$$

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\text{But : } \left(\frac{\partial x}{\partial y} \right)_z = - \frac{\left(\partial z / \partial y \right)_x}{\left(\partial z / \partial x \right)_y} \quad \text{Assuming } dz=0.$$

Note on notation for partial derivatives

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

hold y fixed.

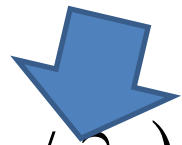
hold x fixed.



Simple change of variables -- continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$



$$\Rightarrow \left(\frac{\partial x}{\partial y} \right)_z = - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y}$$



$$\Rightarrow \left(\frac{\partial x}{\partial z} \right)_y = \frac{1}{(\partial z / \partial x)_y}$$



Simple change of variables -- continued:

Example:

$$z(x, y) = e^{x^2 + y} \quad z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) = (\ln z - y)^{1/2} \quad x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\left(\frac{\partial x}{\partial y} \right)_z \stackrel{?}{=} - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y}$$

$$\left(\frac{\partial x}{\partial z} \right)_y \stackrel{?}{=} \frac{1}{(\partial z / \partial x)_y}$$

$$- \frac{1}{2(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} - \frac{e^{x^2 + y}}{2xe^{x^2 + y}}$$

$$\frac{1}{2z(\ln z - y)^{1/2}} \stackrel{\checkmark}{=} \frac{1}{2xe^{x^2 + y}}$$

Now that we see that these transformations are possible, we should ask the question why we might want to do this?

An example comes from thermodynamics where we have various interdependent variables such as temperature T , pressure P , volume V , etc. etc. Often a measurable property can be specified as a function of two of those, while the other variables are also dependent on those two. For example we might specify T and P while the volume will be $V(T,P)$. Or we might specify T and V while the pressure will be $P(T,V)$.



Other examples from thermo --
For thermodynamic functions:

Internal energy: $U = U(S, V)$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S} \right)_V dS + \left(\frac{\partial U}{\partial V} \right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S} \right)_V \quad P = - \left(\frac{\partial U}{\partial V} \right)_S$$

Enthalpy: $H = H(S, P) = U + PV$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S} \right)_P dS + \left(\frac{\partial H}{\partial P} \right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S} \right)_P \quad V = \left(\frac{\partial H}{\partial P} \right)_S$$



Name	Potential	Differential Form
Internal energy	$E(S, V, N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E, V, N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S, P, N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T, V, N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T, P, N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T, V, \mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$



Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

Let $u \equiv \left(\frac{\partial z}{\partial x} \right)_y$ and $v \equiv \left(\frac{\partial z}{\partial y} \right)_x$

Define new function

$$w(u, y) \Rightarrow dw = \left(\frac{\partial w}{\partial u} \right)_y du + \left(\frac{\partial w}{\partial y} \right)_u dy$$

For $w = z - ux$, $dw = dz - udx - xdu = \cancel{u}dx + vdy - \cancel{u}dx - xdu$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u} \right)_y = -x \quad \left(\frac{\partial w}{\partial y} \right)_u = \left(\frac{\partial z}{\partial y} \right)_x = v$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_{\sigma} \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_{\sigma} \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - L \quad \text{where} \quad p_{\sigma} = \frac{\partial L}{\partial \dot{q}_{\sigma}}$$

$$dH = \sum_{\sigma} \left(\dot{q}_{\sigma} dp_{\sigma} + p_{\sigma} d\dot{q}_{\sigma} - \frac{\partial L}{\partial q_{\sigma}} dq_{\sigma} - \frac{\partial L}{\partial \dot{q}_{\sigma}} d\dot{q}_{\sigma} \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_{\sigma} \left(\frac{\partial H}{\partial q_{\sigma}} dq_{\sigma} + \frac{\partial H}{\partial p_{\sigma}} dp_{\sigma} \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_{\sigma} = \frac{\partial H}{\partial p_{\sigma}} \quad \frac{\partial L}{\partial q_{\sigma}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} \equiv \dot{p}_{\sigma} = -\frac{\partial H}{\partial q_{\sigma}} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$