PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 11-- Chap. 3 & 6 (F &W)

Details and extensions of Lagrangian mechanics

1. Constants of the motion
2. Conserved quantities
3. Legendre transformations

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	#8	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion		

PHYSICS COLLOQUIUM

THURSDAY • SEPTEMBER 16, 2021

Part I
Experimental Projects

This colloquium is designed to give a snapshot of the physics experimental research projects currently in progress at Wake Forest University.

The hope is that these presentations will foster collaborations between groups, inspire beginning students to think about physics research and possibly become engaged in it themselves, and inform more senior students about how their mentors and other mentors approach physics research.

Wake Forest University
Physics Department
Research Opportunities

4PM in Olin 101 and by zoom

More details about HW 7

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Evaluate
   $$\frac{\partial}{\partial q}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial q}.$$

   (b) Evaluate
   $$\frac{\partial}{\partial \dot{q}}\frac{df}{dt} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}}.$$

   (c) Evaluate
   $$\frac{df}{dt}.$$

   (d) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t^2, \quad \text{where} \quad q(t) = \mathrm{e}^{-t/\tau}.$$

   Here τ is a constant. Evaluate df/dt using the expression you just derived. Now find the expression for f as an explicit function of t ($f(t)$) and take its time derivative directly to check your previous results.

More details about HW 7

Suppose that we have a generalized coordinate $q(t)$ that varies with time t and a function that has the dependences $f\left(q(t), \frac{dq}{dt}(t); t\right) \equiv f\left(q(t), \dot{q}(t); t\right)$.

Further comment --

The point of this problem is to fully understand total and partial derivatives

For $W(q(t), \dot{q}(t), t)$, $\frac{dW}{dt} = \frac{\partial W}{\partial q}\dot{q} + \frac{\partial W}{\partial \dot{q}}\ddot{q} + \frac{\partial W}{\partial t}$

Also note that $\frac{\partial^2 W}{\partial q \partial \dot{q}} = \frac{\partial^2 W}{\partial \dot{q} \partial q}$ $\quad \frac{\partial^2 W}{\partial q \partial t} = \frac{\partial^2 W}{\partial t \partial q}$ etc.

But it may NOT be the case that $\frac{\partial}{\partial \dot{q}}\frac{dW}{dt} = \frac{d}{dt}\frac{\partial W}{\partial \dot{q}}$

More details about HW 7

Consider $W(q,\dot{q},t) = q^2\dot{q}^5$ and suppose that $q(t) = t^2$

Simple equation: $W(t) = 32t^9$ $\qquad \frac{dW(t)}{dt} = 288t^8$

Partial derivatives: $\frac{dW(q,\dot{q},t)}{dt} = 2q\dot{q}^6 + 5q^2\dot{q}^4\ddot{q} = ?$

Note that: $$\frac{dW}{dt} = \frac{\partial W}{\partial q}\dot{q} + \frac{\partial W}{\partial \dot{q}}\ddot{q} + \frac{\partial W}{\partial t}$$

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = \text{(constant)}$$

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \qquad \Rightarrow m\dot{x} \equiv p_x \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \qquad \Rightarrow m\dot{y} \equiv p_y \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$

Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

$$\Rightarrow \frac{d}{dt} m r^2 \dot{\varphi} = 0 \qquad \Rightarrow m r^2 \dot{\varphi} \equiv p_\varphi \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{r} = m r \dot{\varphi}^2 - \frac{\partial V}{\partial r} = \frac{{p_\varphi}^2}{m r^3} - \frac{\partial V}{\partial r}$$

Starting from:

Recall alternative form of Euler-Lagrange equations:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that:

$$\frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \sum_\sigma \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t}$$

Additional constant of the motion:

If $\frac{\partial L}{\partial t} = 0;$

then: $$\frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(z)\right) = -E \quad \text{(constant)}$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2}m\dot{z}^2 + V(z)$$

Summary from previous slide

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

→3 variable functions

$$E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2} m\dot{z}^2 + V(z)$$

p_x, p_y, E constant

→1 variable function

Why might this be useful?

Additional constant of the motion -- continued:

$$\text{If } \frac{\partial L}{\partial t} = 0;$$

$$\text{then: } \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 2: Motion in a central potential

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$ ➔2 variable functions

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r) - m\dot{r}^2 - mr^2\dot{\phi}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)\right) = -E \quad \text{(constant)}$$

For this case, we also have $mr^2\dot{\varphi} \equiv p_\varphi$

$$\Rightarrow E = \frac{p_\varphi^2}{2mr^2} + \tfrac{1}{2}m\dot{r}^2 + V(r)$$ ➔1 variable function

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$- \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$\Rightarrow E = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{p_z^2}{2m}$$

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$\frac{\partial L}{\partial x} = 0 \qquad \Rightarrow m\dot{x} = p_x \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$- \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c} B_0 \dot{x} y$$

$$\Rightarrow E = \tfrac{1}{2} m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x,y) \quad \Leftrightarrow \quad x(y,z)???$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\text{But}: \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\partial z / \partial y\right)_x}{\left(\partial z / \partial x\right)_y}$$ Assuming $dz=0$.

Note on notation for partial derivatives

$$z(x,y) \quad \Rightarrow \quad dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

hold *y* fixed.

hold *x* fixed.

Simple change of variables -- continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\partial z / \partial y\right)_x}{\left(\partial z / \partial x\right)_y} \qquad \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{\left(\partial z / \partial x\right)_y}$$

Simple change of variables -- continued:

Example:

$$z(x,y) = e^{x^2+y}$$

$$x(y,z) = \left(\ln z - y\right)^{1/2}$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\left(\frac{\partial x}{\partial y}\right)_z \stackrel{?}{=} -\frac{\left(\partial z/\partial y\right)_x}{\left(\partial z/\partial x\right)_y} \qquad \left(\frac{\partial x}{\partial z}\right)_y \stackrel{?}{=} \frac{1}{\left(\partial z/\partial x\right)_y}$$

$$-\frac{1}{2\left(\ln z - y\right)^{1/2}} \stackrel{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \qquad \frac{1}{2z\left(\ln z - y\right)^{1/2}} \stackrel{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

Now that we see that these transformations are possible, we should ask the question why we might want to do this?

An example comes from thermodynamics where we have various interdependent variables such as temperature T, pressure P, volume V, etc. etc. Often a measurable property can be specified as a function of two of those, while the other variables are also dependent on those two. For example we might specify T and P while the volume will be V(T,P). Or we might specify T and V while the pressure will be P(T,V).

Other examples from thermo --
For thermodynamic functions:

$$\text{Internal energy}: \quad U = U(S,V)$$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

$$\text{Enthalpy}: \quad H = H(S,P) = U + PV$$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \qquad V = \left(\frac{\partial H}{\partial P}\right)_S$$

Name	Potential	Differential Form
Internal energy	$E(S,V,N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E,V,N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S,P,N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T,V,N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T,P,N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T,V,\mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Let $u \equiv \left(\frac{\partial z}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial z}{\partial y}\right)_x$

Define new function

$$w(u,y) \Rightarrow dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

For $w = z - ux,$ $\quad dw = dz - udx - xdu = udx + vdy - udx - xdu$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \qquad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$