



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Notes for Lecture 12 – Chap. 3&6 (F&W)

- 1. Constructing the Hamiltonian**
- 2. Hamilton's canonical equation**
- 3. Examples**

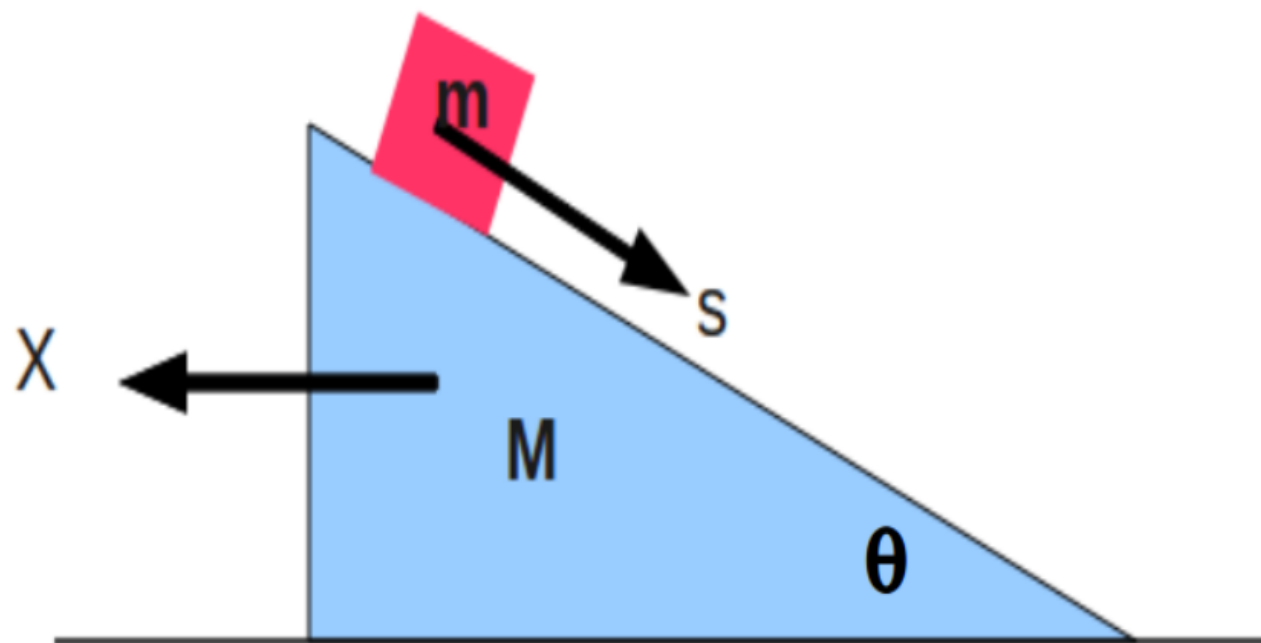
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	#8	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/20/2021



Continue reading Chapters 3 and 6 in **Fetter and Walecka**.



1. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ , m , M , etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ .)

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

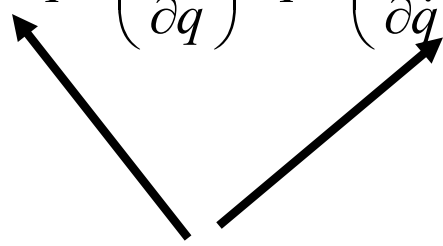
Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_{\sigma} \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_{\sigma} \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Application of the Legendre transformation for the Lagrangian and Hamiltonian
 $L(q, \dot{q}, t)$ and $H(q, p, t)$

suppose $H(q, p, t) = \dot{q}p - L(q, \dot{q}, t)$

$$dH = \dot{q}dp + p d\dot{q} - \left(\frac{\partial L}{\partial q}\right) dq - \left(\frac{\partial L}{\partial \dot{q}}\right) d\dot{q} - \left(\frac{\partial L}{\partial t}\right) dt = \left(\frac{\partial H}{\partial q}\right) dq + \left(\frac{\partial H}{\partial p}\right) dp + \left(\frac{\partial H}{\partial t}\right) dt$$


Note that these two terms cancel if $p = \frac{\partial L}{\partial \dot{q}}$

The analysis on the following slides is a generalization to multiple dimensions q_σ and p_σ



Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - L \quad \text{where} \quad p_{\sigma} = \frac{\partial L}{\partial \dot{q}_{\sigma}}$$

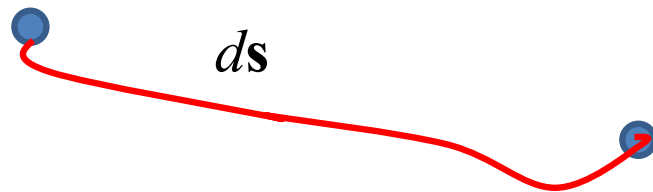
$$\begin{aligned} dH &= \sum_{\sigma} \left(\dot{q}_{\sigma} dp_{\sigma} + \cancel{p_{\sigma} d\dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} dq_{\sigma} - \cancel{\frac{\partial L}{\partial \dot{q}_{\sigma}} d\dot{q}_{\sigma}} \right) - \frac{\partial L}{\partial t} dt \\ &= \sum_{\sigma} \left(\dot{q}_{\sigma} dp_{\sigma} - \frac{\partial L}{\partial q_{\sigma}} dq_{\sigma} \right) - \frac{\partial L}{\partial t} dt \end{aligned}$$

$$dH = \sum_{\sigma} \left(\frac{\partial H}{\partial q_{\sigma}} dq_{\sigma} + \frac{\partial H}{\partial p_{\sigma}} dp_{\sigma} \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_{\sigma} = \frac{\partial H}{\partial p_{\sigma}} \quad \frac{\partial L}{\partial q_{\sigma}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} \equiv \dot{p}_{\sigma} = -\frac{\partial H}{\partial q_{\sigma}} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$



Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates :
 $q_{\sigma}(\{x_i\})$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_{\sigma}\}, \{\dot{q}_{\sigma}\}, t)$$

$$\Rightarrow \text{Minimization integral: } S = \int_{t_i}^{t_f} L(\{q_{\sigma}\}, \{\dot{q}_{\sigma}\}, t) dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_{\sigma}(t)\}, \{p_{\sigma}(t)\}, t)$$

$$H = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - L \quad \Rightarrow \quad L = \sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - H(\{q_{\sigma}(t)\}, \{p_{\sigma}(t)\}, t)$$

Hamilton's principle continued: Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_{\sigma} \dot{q}_{\sigma} p_{\sigma} - H(\{q_{\sigma}(t)\}, \{p_{\sigma}(t)\}, t) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_{\sigma} \left(\dot{q}_{\sigma} \delta p_{\sigma} + \delta \dot{q}_{\sigma} p_{\sigma} - \frac{\partial H}{\partial q_{\sigma}} \delta q_{\sigma} - \frac{\partial H}{\partial p_{\sigma}} \delta p_{\sigma} \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_{\sigma} = \frac{\partial H}{\partial p_{\sigma}}$$

$$\Rightarrow \dot{p}_{\sigma} = -\frac{\partial H}{\partial q_{\sigma}}$$

Canonical equations

Detail:

$$\int_{t_i}^{t_f} \left(\sum_{\sigma} (\delta \dot{q}_{\sigma} p_{\sigma}) \right) dt = \int_{t_i}^{t_f} \left(\sum_{\sigma} \left(\frac{d(\delta q_{\sigma} p_{\sigma})}{dt} - \delta q_{\sigma} \dot{p}_{\sigma} \right) \right) dt = \sum_{\sigma} \delta q_{\sigma} p_{\sigma} \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \left(\sum_{\sigma} (\delta q_{\sigma} \dot{p}_{\sigma}) \right) dt$$

Note: In the original image, a blue arrow points from the term $\sum_{\sigma} \delta q_{\sigma} p_{\sigma} \Big|_{t_i}^{t_f}$ to a '0' at the top right, indicating it vanishes due to boundary conditions.

More comments about “details”

Detail:

$$\int_{t_i}^{t_f} \left(\sum_{\sigma} (\delta \dot{q}_{\sigma} p_{\sigma}) \right) dt = \int_{t_i}^{t_f} \left(\sum_{\sigma} \left(\frac{d(\delta q_{\sigma} p_{\sigma})}{dt} - \delta q_{\sigma} \dot{p}_{\sigma} \right) \right) dt = \underbrace{\sum_{\sigma} \delta q_{\sigma} p_{\sigma} \Big|_{t_i}^{t_f}}_{\text{Vanishes because } \delta q(t_f) = \delta q(t_i) \text{ due to the premise of Hamilton's principle.}} - \int_{t_i}^{t_f} \left(\sum_{\sigma} (\delta q_{\sigma} \dot{p}_{\sigma}) \right) dt$$

Vanishes because
 $\delta q(t_f) = \delta q(t_i)$ due to
the premise of
Hamilton's principle.

In the Hamiltonian formulation --

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Why are these equations known as the “canonical equations”?

- a. Because they are beautiful.
- b. The term is meant to elevate their importance to the level of the music of J. S. Bach
- c. To help you remember them
- d. No good reason; it is just a name



Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

What happens when you miss a step in the recipe?

- a. No big deal
- b. Big deal – can lead to shame and humiliation
(or at least wrong analysis)



Constants of the motion in Hamiltonian formalism

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma (-\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

What is the physical meaning to a constant H ?

Comment -- Whenever you find a constant of the motion, it is helpful for analyzing the trajectory. In this case, H often represents the mechanical energy of the system so that constant H implies that energy is conserved.

Example 1: one-dimensional potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$p_x = m\dot{x} \quad p_y = m\dot{y} \quad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants: $\bar{p}_x, \bar{p}_y, \bar{H}$ (using bar to indicate constant)

Equations of motion:

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \frac{dp_z}{dt} = - \frac{dV}{dz}$$



Example 2: Motion in a central potential

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r)$$

$$p_r = m\dot{r} \qquad p_\phi = mr^2\dot{\phi}$$

$$\begin{aligned} H &= m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r) \right) \\ &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + V(r) \end{aligned}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants: $\bar{p}_\phi, \bar{H}$

Equations of motion:

$$\frac{dr}{dt} = \frac{p_r}{m} \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$



Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right)^2 - Mgh \cos \beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2 \beta + I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right) \cos \beta$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right)$$

$$H = \frac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\alpha} \cos \beta + \dot{\gamma} \right)^2 + Mgh \cos \beta$$

$$H = \frac{\left(p_\alpha - p_\gamma \cos \beta \right)^2}{2I_1 \sin^2 \beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh \cos \beta$$

Constants: $\bar{p}_\alpha, \bar{p}_\gamma, \bar{H}$



Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants: $\bar{p}_z, \bar{H}$



Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants: $\bar{p}_z, \bar{H}$

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc} \left(p_y - \frac{q}{2c} B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc} \left(p_x + \frac{q}{2c} B_0 y\right)$$



Canonical equations of motion for constant magnetic field

-- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc} \left(p_y - \frac{q}{2c} B_0 x \right) = \frac{qB_0}{2c} \frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc} \left(p_x + \frac{q}{2c} B_0 y \right) = -\frac{qB_0}{2c} \frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc} B_0 \dot{y} = \frac{qB_0}{mc} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc} B_0 \dot{x} = -\frac{qB_0}{mc} \frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{qB_0}{mc} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = -\frac{qB_0}{mc} \frac{dx}{dt}$$

Are these results equivalent to the results of the Lagrangian analysis?

- a. Yes
- b. No



General treatment of particle of mass m and charge q moving in 3 dimensions in an potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$:

Lagrangian:
$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Hamiltonian:
$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t)$$

$$\begin{aligned} H(\mathbf{r}, \mathbf{p}, t) &= \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}, t) \\ &= \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right)^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t) \end{aligned}$$



Some details: $L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

Hamiltonian: $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m \dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t)$

$$H(\mathbf{r}, \mathbf{p}, t) = \mathbf{p} \cdot \dot{\mathbf{r}} - L(\mathbf{r}, \dot{\mathbf{r}}, t)$$

$$= \left(m \dot{\mathbf{r}} + \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right) \cdot \dot{\mathbf{r}} - \left(\frac{1}{2} m \dot{\mathbf{r}}^2 - U(\mathbf{r}) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) \right)$$

$$= \frac{1}{2} m \dot{\mathbf{r}}^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t)$$

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r}, t) \right)^2 + U(\mathbf{r}) + q\Phi(\mathbf{r}, t)$$



Canonical form



Other properties of Hamiltonian formalism – Poisson brackets:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

For an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \dot{q}_{\sigma} + \frac{\partial F}{\partial p_{\sigma}} \dot{p}_{\sigma} \right) + \frac{\partial F}{\partial t} = \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \frac{\partial H}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial H}{\partial q_{\sigma}} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F, G]_{PB} \equiv \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \frac{\partial G}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial G}{\partial q_{\sigma}} \right) = -[G, F]_{PB}$$

So that :
$$\frac{dF}{dt} = [F, H]_{PB} + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

$$[F, G]_{PB} \equiv \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \frac{\partial G}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial G}{\partial q_{\sigma}} \right) = -[G, F]_{PB}$$

Examples:

$$\begin{aligned} [x, x]_{PB} &= 0 & [x, p_x]_{PB} &= 1 & [x, p_y]_{PB} &= 0 \\ [L_x, L_y]_{PB} &= L_z \end{aligned}$$

Liouville theorem

Let $D \equiv$ density of particles in phase space:

$$\frac{dD}{dt} = [D, H]_{PB} + \frac{\partial D}{\partial t} = 0 \quad \longrightarrow \quad \text{For next time}$$