PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 18: Chap. 7 (F&W)

Mechanical motion of a continuous string

1. Comments on linear vs. non-linear differential equations – considering beyond harmonic oscillations
2. Back to linear analyses -- masses coupled by springs ←→mass continuum coupled by string
3. Mechanics one-dimensional continuous system
4. The wave equation

The one dimensional motion of a large number masses interconnected with springs provides a model of longitudinal motions of a continuous elastic spring and related topics covered in Chapter 7 of your textbook

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	#8	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/20/2021
13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorm	#10	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021

Start reading Chapter 7. The homework problem concerns one dimensional wave motion using methods discussed in this lecture.

PHY 711 – Assignment #14

10/01/2021

Start reading Chapter 7 in **Fetter and Walecka**.

Consider a one-dimensional wave characterized by displacement $\mu(x,t)$ as a function of position x and time t is described by the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0, \qquad (1)$$

where c denotes the wave speed. Find the functional form of $\mu(x,t)$ for each of these initial conditions.

1. At $t = 0$,

$$\mu(x,0) = \frac{A}{\cosh(x)} \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = 0, \qquad (2)$$

   where A is a given amplitude.

2. At $t = 0$,

$$\mu(x,0) = 0 \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = \frac{A \sinh(x)}{\cosh^2(x)}. \qquad (3)$$

Homework problem is due Wednesday.

Schedule

October 2021

S	M	T	W	T	F	S
26	27	28	29	30	1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

Fall break (7–10)

Take-home exam (11–15)

Digression – comment on linear vs non-linear equations

Linear oscillator equations (ODE example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

Euler-Lagrange equations: $\ddot{x} = -\omega^2 x$

Superposition property of linear equations: --

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$\Rightarrow Ax_1(t) + Bx_2(t)$ are also solutions (all A, B)

Digression on the special properties of linear equations in contrast to complications for non-linear equations.

Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left.\frac{d^4V}{dx^4}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

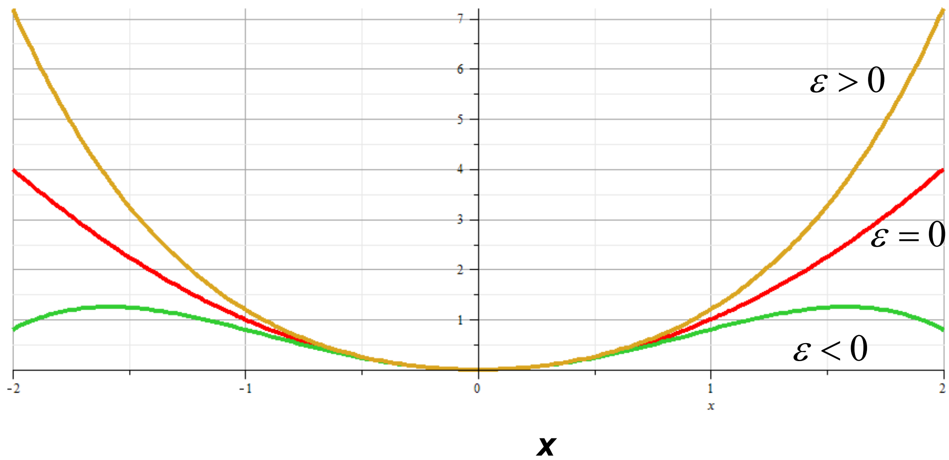
Euler - Lagrange equations :

$$\ddot{x} = -\omega^2(x + \varepsilon x^3)$$

Superposition-- no longer applies

An example of the effects of non-linearity.

$$V(x) \approx \frac{1}{2} m\omega^2 \left(x^2 + \varepsilon x^4\right)$$



Plot of nonlinear potentials.

Non-linear example -- continued

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler-Lagrange equations:

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler-Lagrange equations:

zero order: $\ddot{x}_0 + \omega^2 x_0 = 0$

first order: $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

Approximate solution to example non-linear equation.

Non-linear example -- continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions: $x(0) = X_0 \qquad \dot{x}(0) = 0$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler-Lagrange equations:

zero order: $\ddot{x}_0 + \omega^2 x_0 = 0 \qquad \Rightarrow x_0(t) = X_0 \cos(\omega t)$

first order: $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

$$\Rightarrow \ddot{x}_1(t) + \omega^2 x_1(t) = -X_0^3 \cos^3(\omega t) = -\frac{X_0^3}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right)$$

$$\Rightarrow x_1(t) = -\frac{X_0^3}{8\omega^2}\left\{3\omega t \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\}$$

$$x(t) = X_0 \cos(\omega t) - \varepsilon\frac{X_0^3}{8\omega^2}\left\{3\omega t \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\} + O\left(\varepsilon^2\right)$$

Non-linear equation continued.

Non-linear example -- continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions:

$$x(0) = X_0 \qquad \dot{x}(0) = 0$$

Perturbation expansion:

$$x\left(t\right) = x_0\left(t\right) + \varepsilon x_1\left(t\right) + \ldots$$

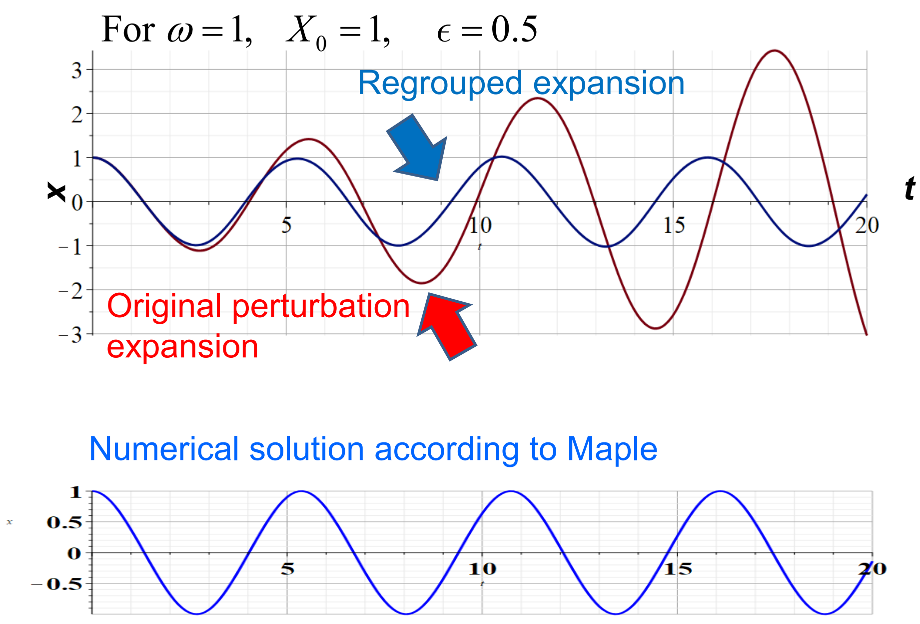
Previous result (blows up at large t):

$$x(t) = X_0 \cos\left(\omega t\right) - \varepsilon \frac{X_0^3}{8\omega^2} \left\{3\omega t \ \sin\left(\omega t\right) + \frac{1}{4}\left[\cos\left(\omega t\right) - \cos\left(3\omega t\right)\right]\right\} + O\left(\varepsilon^2\right)$$

By rearranging terms (allowing effective frequency to vary):

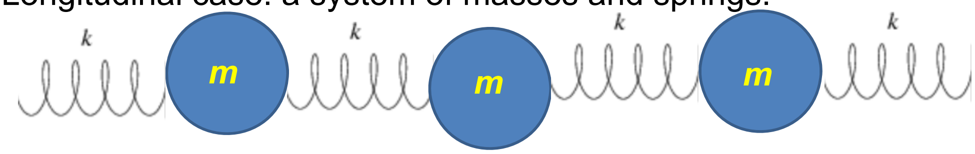
$$x(t) = X_0 \cos\left(\omega\left(1 + \varepsilon \frac{3X_0^2}{8\omega}\right)t\right) - \varepsilon \frac{X_0^3}{32\omega^2} \left\{\cos\left(\omega t\right) - \cos\left(3\omega t\right)\right\} + O\left(\varepsilon^2\right)$$

More details.



Plot of results.

Back to linear equations –
Longitudinal case: a system of masses and springs:



$$x_{i-1} \qquad x_i \qquad x_{i+1}$$

$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - x_i\right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x,t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} \left(\Delta x\right)^2$$

Showing how the case of the infinite mass and spring system approximates the continuous elastic string.

Discrete equation: $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation: $m\dfrac{\partial^2 \mu}{\partial t^2} = k\left(\Delta x\right)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m/\Delta x}\right)\frac{\partial^2 \mu}{\partial x^2}$$

system parameter with units of (velocity)^2

For transverse oscillations on a string with tension τ and mass/length σ:

$$\left(\frac{k\Delta x}{m/\Delta x}\right) \Rightarrow \frac{\tau}{\sigma}$$

Regrouping constants in terms of spring constant times increment of length and mass per unit length which combine to give a squared velocity for the longitudinal case. For the transverse case, string tension is involved.

More details

Longitudinal case

Consider Taylor's series (focussing on x-dependence)

$$\mu(x+\Delta x) = \mu(x) + \Delta x \left.\frac{d\mu}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2\mu}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^3 \left.\frac{d^3\mu}{dx^3}\right|_x + \frac{1}{24}(\Delta x)^4 \left.\frac{d^4\mu}{dx^4}\right|_x + ...$$

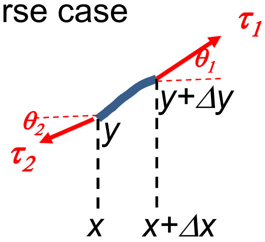
$$\mu(x-\Delta x) = \mu(x) - \Delta x \left.\frac{d\mu}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2\mu}{dx^2}\right|_x - \frac{1}{6}(\Delta x)^3 \left.\frac{d^3\mu}{dx^3}\right|_x + \frac{1}{24}(\Delta x)^4 \left.\frac{d^4\mu}{dx^4}\right|_x + ...$$

Therefore $(\Delta x)^2 \left.\frac{d^2\mu}{dx^2}\right|_x = \mu(x+\Delta x) + \mu(x-\Delta x) - 2\mu(x) - \frac{1}{12}(\Delta x)^4 \left.\frac{d^4\mu}{dx^4}\right|_x + ...$

$$\Rightarrow \left.\frac{d^2\mu}{dx^2}\right|_x \approx \frac{\mu(x+\Delta x) + \mu(x-\Delta x) - 2\mu(x)}{(\Delta x)^2}$$

More details

Transverse case



Net vertical force on increment of string:

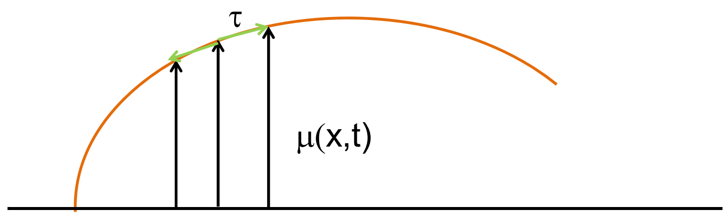
$$\tau_1 \sin\theta_1 - \tau_2 \sin\theta_2 \approx \tau_1 \tan\theta_1 - \tau_2 \tan\theta_2$$

$$\approx \tau\left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x} \right)$$

$$= \tau \Delta x \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \tau\left(\Delta x \frac{d^2 y}{dx^2} \right)$$

Transverse displacement:



$$\frac{\partial^2 \mu}{\partial t^2} = \frac{\tau}{\sigma}\frac{\partial^2 \mu}{\partial x^2}$$

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

The diagram shows how the y component of the net tension contributes to the transverse motion.

Lagrangian for continuous system:

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle:

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx\, L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

It is possible to adapt the Lagrangian formalism to this continuous system.

Euler - Lagrange equations for continuous system:

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Example:

$$L = \frac{\sigma}{2}\left(\frac{\partial \mu}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial \mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$

The continuum version of the Euler-Lagrange equations result in the wave equation for this example.

Note that this is an example of a partial differential equation

General solutions $\mu(x,t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

In the next several slides we will discuss solutions to the wave equation. Note that the one dimensional wave equation has some special properties.

Initial value solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

These functions would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

This method by D’Alembert is based on the special property of the wave equation.

Solution -- continued : $\mu(x,t) = f(x-ct) + g(x+ct)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

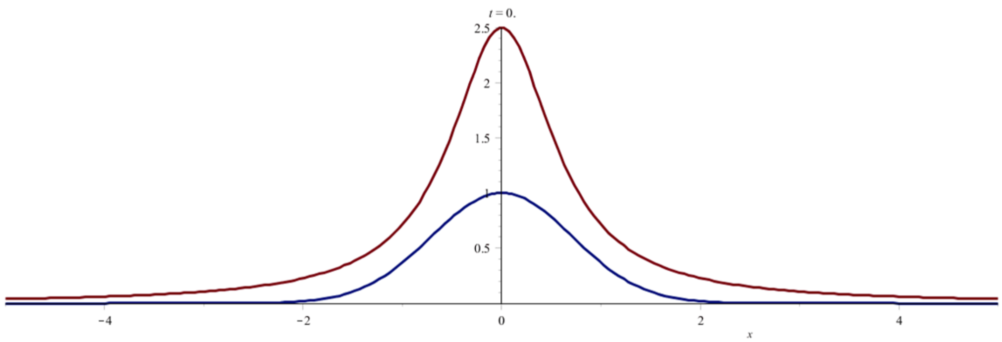
$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

D’Alembert’s method continued.

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x,t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$



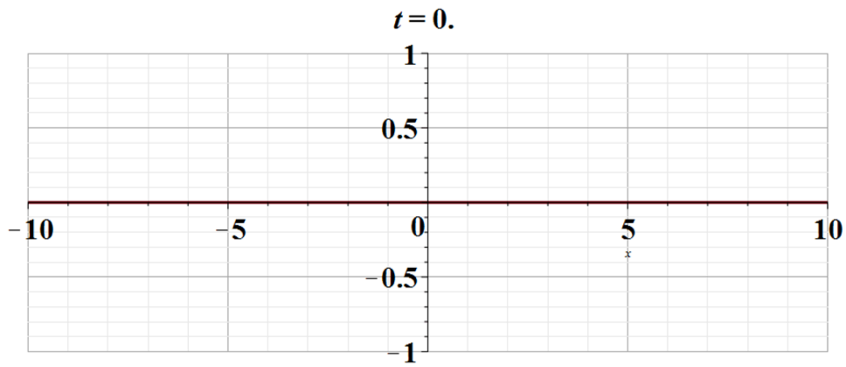
An example. (Use slide show to see animation.)

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $$\frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$$



Another example. Use slide show to see animation.