

**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF in Olin 103**

**Notes for Lecture 19 – Chap. 7 (F&W)**

**Solutions of differential equations**

- 1. The wave equation**
- 2. Sturm-Liouville equation**
- 3. Green's function solution methods**

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In this lecture, we follow the textbook to use the example of the one-dimensional wave equation to discuss ordinary differential equations more generally and develop some solution methods.

4	Mon, 8/30/2021	Chap. 1	Scattering theory	<a href="#">#3</a>	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	<a href="#">#4</a>	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	<a href="#">#5</a>	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	<a href="#">#6</a>	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	<a href="#">#7</a>	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	<a href="#">#8</a>	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	<a href="#">#9</a>	9/20/2021
13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorem	<a href="#">#10</a>	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	<a href="#">#11</a>	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	<a href="#">#12</a>	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	<a href="#">#13</a>	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	<a href="#">#14</a>	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		

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Expected schedule for the next weeks...

One-dimensional wave equation  
representing longitudinal or transverse displacements  
as a function of  $x$  and  $t$ , an example of a partial  
differential equation --

For the displacement function,  $\mu(x,t)$ , the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function  $f(q)$  or  $g(q)$ :

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Review of wave equation.

## The wave equation and related linear PDE's

One dimensional wave equation for  $\mu(x,t)$ :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x) \frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left( \tau(x) \frac{\partial \mu(x,t)}{\partial x} \right) + v(x) \mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \phi(x) \cos(\omega t + \alpha)$$

Sturm-Liouville equation for spatial function  $\phi(x)$ :

$$-\frac{d}{dx} \left( \tau(x) \frac{d\phi(x)}{dx} \right) + v(x) \phi(x) = \omega^2 \sigma(x) \phi(x)$$

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Generalization of the wave equation. Equations in this class are separable in the time variables and the spatial variable satisfies a generalized eigenvalue problem of this form.

## Linear second-order ordinary differential equations Sturm-Liouville equations

Inhomogenous problem:  $\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$

given functions

solution to be determined

applied force

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem:  $F(x)=0$

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We will sometimes want to generalize even further with an “inhomogeneous” term such as an applied force.

Examples of Sturm-Liouville eigenvalue equations --

$$\left( -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) \varphi(x) = 0$$

Bessel functions:  $0 \leq x \leq \infty$

$$\tau(x) = -x \quad v(x) = x \quad \sigma(x) = \frac{1}{x} \quad \lambda = \nu^2 \quad \varphi(x) = J_\nu(x)$$

Legendre functions:  $-1 \leq x \leq 1$

$$\tau(x) = -(1-x^2) \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = l(l+1) \quad \varphi(x) = P_l(x)$$

Fourier functions:  $0 \leq x \leq 1$

$$\tau(x) = 1 \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = n^2\pi^2 \quad \varphi(x) = \sin(n\pi x)$$

For now, we will focus on eigenvalues of the homogeneous equations.

### Solution methods of Sturm-Liouville equations

(assume all functions and constants are real):

$$\text{Homogenous problem: } \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi_0(x) = 0$$

$$\text{Inhomogenous problem: } \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$$

Eigenfunctions :

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

$$\text{Orthogonality of eigenfunctions: } \int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n,$$

$$\text{where } N_n \equiv \int_a^b \sigma(x) (f_n(x))^2 dx.$$

Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

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$N_n$

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The eigenfunctions of these equations have very useful properties such as completeness.

Why all of the fuss about eigenvalues and eigenvectors?

- a. They are always necessary for solving differential equations
- b. Not all eigenfunctions have analytic forms.
- c. It is possible to solve a differential equation without the use of eigenfunctions.
- d. Eigenfunctions have some useful properties.



### Comment on orthogonality of eigenfunctions

$$\begin{aligned}
 \left( -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_n(x) &= \lambda_n \sigma(x) f_n(x) \\
 \left( -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_m(x) &= \lambda_m \sigma(x) f_m(x) \\
 f_m(x) \left( -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_n(x) - f_n(x) \left( -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_m(x) \\
 &= (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x) \\
 -\frac{d}{dx} \left( f_m(x) \tau(x) \frac{df_n(x)}{dx} - f_n(x) \tau(x) \frac{df_m(x)}{dx} \right) &= (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x)
 \end{aligned}$$

Orthogonality of eigenfunctions.

### Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=(\lambda_n-\lambda_m)\sigma(x)f_n(x)f_m(x)$$

Now consider integrating both sides of the equation in the interval

$$a \leq x \leq b :$$

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)\bigg|_a^b=(\lambda_n-\lambda_m)\int_a^b dx\sigma(x)f_n(x)f_m(x)$$



Vanishes for various boundary conditions  
at  $x=a$  and  $x=b$

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Orthogonality continued.

Comment on orthogonality of eigenfunctions -- continued

$$-\left( f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) \Big|_a^b = (\lambda_n - \lambda_m) \int_a^b dx \sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1.  $f_m(a) = f_m(b) = 0$

2.  $\tau(x) \frac{df_m(x)}{dx} \Big|_a = \tau(x) \frac{df_m(x)}{dx} \Big|_b = 0$

3.  $f_m(a) = f_m(b)$  and  $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

$$\int_a^b dx \sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

Orthogonality continued.

### Comment on "completeness"

It can be shown that for any reasonable function  $h(x)$ , defined within the interval  $a < x < b$ , we can expand that function as a linear combination of the eigenfunctions  $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where } C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

These ideas lead to the notion that the set of eigenfunctions  $f_n(x)$  form a "complete" set in the sense of "spanning" the space of all functions in the interval  $a < x < b$ , as summarized by the statement:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x').$$

Notion of completeness.

Comment on “completeness” -- continued

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where } C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx \sigma(x) \left( h(x) - \sum_n C_n f_n(x) \right)^2$$

$\epsilon^2$  can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2 \int_a^b dx \sigma(x) \left( h(x) - \sum_n C_n f_n(x) \right) f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m} \int_a^b dx \sigma(x) h(x) f_m(x)$$

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Notion of completeness and practical applications.

### Variational approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues  $\lambda_n$  and eigenfunctions  $f_n(x)$ . When it is not possible to find the "exact" functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \quad S(x) \equiv -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x)$$

where  $\tilde{h}(x)$  is a variable function which satisfies the correct boundary values. The "proof" of this inequality is based on the notion that  $\tilde{h}(x)$  can in principle be expanded in terms of the (unknown) exact eigenfunctions  $f_n(x)$ :

$$\tilde{h}(x) = \sum_n C_n f_n(x), \quad \text{where the coefficients } C_n \text{ can be}$$

assumed to be real.

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A very useful property of eigenfunctions related to homework problem

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x) \sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h} | S | \tilde{h} \rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h} | \sigma | \tilde{h} \rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

$$\text{Therefore } \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

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Proof of theorem continued.

### Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

Example:  $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{with } f_n(0) = f_n(a) = 0$   
trial function  $f_{\text{trial}}(x) = x(x-a)$

Exact value of  $\lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Raleigh-Ritz estimate:  $\frac{\langle x(a-x) | -\frac{d^2}{dx^2} | x(a-x) \rangle}{\langle x(a-x) | x(a-x) \rangle} = \frac{10}{a^2}$

Example of the Rayleigh Ritz method.



A generally useful solution method -- Green's function approach

Suppose that we can find a Green's function defined as follows:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Completeness of eigenfunctions:

Recall:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

In terms of eigenfunctions:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x, x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda}$$

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The following slides present solution methods for differential equations involving the use of eigenvalues.

Solution to inhomogeneous problem by using Green's functions

Inhomogeneous problem:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = F(x)$$

Green's function :

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_{\lambda}(x, x') = \delta(x - x')$$

Formal solution:

$$\varphi_{\lambda}(x) = \varphi_{\lambda 0}(x) + \int_0^L G_{\lambda}(x, x') F(x') dx'$$

Solution to homogeneous problem

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From a knowledge of the Green's function we can find solutions of related inhomogeneous equations.

Example Sturm-Liouville problem:

Example:  $\tau(x) = 1$ ;  $\sigma(x) = 1$ ;  $v(x) = 0$ ;  $a = 0$  and  $b = L$

$$\lambda = 1; \quad F(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Example.

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x) \sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

In this example: 
$$\frac{2}{L} \sum_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x-x')$$

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Solution using eigenfunctions.

Green's function :

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_{\lambda}(x, x') = \delta(x - x')$$

Green's function for the example :

$$G(x, x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda} = \frac{2}{L} \sum_n \frac{\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}$$

Continued.

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

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In this case, the solution simplifies.

Alternate Green's function method :

$$G(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

$$\left(-\frac{d^2}{dx^2} - 1\right) g_i(x) = 0 \quad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L-x);$$

$$W = g_b(x) \frac{dg_a(x)}{dx} - g_a(x) \frac{dg_b(x)}{dx} = \sin(L-x)\cos(x) + \sin(x)\cos(L-x) \\ = \sin(L)$$

$$\phi(x) = \phi_0(x) + \frac{\sin(L-x)}{\sin(L)} \int_0^x \sin(x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx' \\ + \frac{\sin(x)}{\sin(L)} \int_x^L \sin(L-x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\phi(x) = \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

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Another method of finding a Green's function.

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Two homogeneous solutions

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) g_i(x) = 0 \quad \text{for } i = a, b$$

Let

$$G_\lambda(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

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Green's function based on homogeneous solutions (not eigenfunctions).



For  $\epsilon \rightarrow 0$ :

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \int_{x'-\epsilon}^{x'+\epsilon} dx \delta(x - x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} \right) \frac{1}{W} g_a(x_<) g_b(x_>) = 1$$

$$-\frac{\tau(x)}{W} \left( \frac{d}{dx} g_a(x_<) g_b(x_>) \right) \Big|_{x'-\epsilon}^{x'+\epsilon} = \frac{\tau(x')}{W} \left( g_a(x') \frac{d}{dx} g_b(x') - g_b(x') \frac{d}{dx} g_a(x') \right)$$

$$\Rightarrow W = \tau(x') \left( g_a(x') \frac{d}{dx} g_b(x') - g_b(x') \frac{d}{dx} g_a(x') \right)$$

Note --  $W$  (Wronskian) is constant, since  $\frac{dW}{dx'} = 0$ .

$\Rightarrow$  Useful Green's function construction in one dimension:

$$G_\lambda(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

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Some details.

$$\left( -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) \varphi(x) = F(x)$$

Green's function solution:

$$\begin{aligned} \varphi_\lambda(x) &= \varphi_{\lambda 0}(x) + \int_{x_l}^{x_u} G_\lambda(x, x') F(x') dx' \\ &= \varphi_{\lambda 0}(x) + \frac{g_b(x)}{W} \int_{x_l}^x g_a(x') F(x') dx' + \frac{g_a(x)}{W} \int_x^{x_u} g_b(x') F(x') dx' \end{aligned}$$

More details. To be continued.