PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion on Lecture 19 – Chap. 7 (F&W)

Solutions of differential equations

1. **The wave equation**
2. **Sturm-Liouville equation**
3. **Green's function solution methods**

In this lecture, we follow the textbook to use the example of the one-dimensional wave equation to discuss ordinary differential equations more generally and develop some solution methods.

4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	#8	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/20/2021
13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorm	#10	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		

Expected schedule for the next weeks…

Your questions –
From Owen -- What is the benefit of using Green's functions to solve differential equations compared to other methods?

Comment -- In this lecture and when we resume next week (10/15/2021) we will learn/review a number of solution methods, each having their particular strengths and weaknesses….

Please send me your preferences for review topics for Wednesday's lecture ASAP.

One-dimensional wave equation

representing longitudinal or transverse displacements as a function of x and t, an example of a partial differential equation --

For the displacement function, $\mu(x,t)$, the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Review of wave equation.

The wave equation and related linear PDE's

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x)\frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial \mu(x,t)}{\partial x}\right) + v(x)\mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \phi(x)\cos(\omega t + \alpha)$$

Sturm-Liouville equation for spatial function $\phi(x)$:

$$-\frac{d}{dx}\left(\tau(x)\frac{d\phi(x)}{dx}\right) + v(x)\phi(x) = \omega^2\sigma(x)\phi(x)$$

Generalization of the wave equation. Equations in this class are separable in the time variables and the spatial variable satisfies a generalized eigenvalue problem of this form.

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

given functions

applied force

solution to be determined

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem: *F(x)=0*

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We will sometimes want to generalize even further with an “inhomogeneous” term such as an applied force.

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=0$$

Bessel functions: $0 \le x \le \infty$

$$\tau(x)=-x \quad v(x)=x \quad \sigma(x)=\frac{1}{x} \quad \lambda=\nu^2 \quad \varphi(x)=J_\nu(x)$$

Legendre functions: $-1 \le x \le 1$

$$\tau(x)=-\left(1-x^2\right) \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=l(l+1) \quad \varphi(x)=P_l(x)$$

Fourier functions: $0 \le x \le 1$

$$\tau(x)=1 \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=n^2\pi^2 \quad \varphi(x)=\sin(n\pi x)$$

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For now, we will focus on eigenvalues of the homogeneous equations.

Solution methods of Sturm-Liouville equations
(assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n\equiv\int_a^b\sigma(x)(f_n(x))^2\,dx.$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

The eigenfunctions of these equations have very useful properties such as completeness.

Why all of the fuss about eigenvalues and eigenvectors?

a. They are sometimes useful in finding solutions to differential equations
b. Not all eigenfunctions have analytic forms.
c. It is possible to solve a differential equation without the use of eigenfunctions.
d. Eigenfunctions have some useful properties.

Comment on orthogonality of eigenfunctions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)=\lambda_m\sigma(x)f_m(x)$$

$$f_m(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)-f_n(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)$$
$$=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

Orthogonality of eigenfunctions.

Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) = \left(\lambda_n - \lambda_m\right)\sigma(x) f_n(x) f_m(x)$$

Now consider integrating both sides of the equation in the interval $a \le x \le b$:

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\,\sigma(x) f_n(x) f_m(x)$$

Vanishes for various boundary conditions at *x=a and x=b*

Orthogonality continued.

Comment on orthogonality of eigenfunctions -- continued

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1. $f_m(a) = f_m(b) = 0$

2. $\tau(x)\frac{df_m(x)}{dx}\Big|_a = \tau(x)\frac{df_m(x)}{dx}\Big|_b = 0$

3. $f_m(a) = f_m(b)$ and $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

$$\int_a^b dx\sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

Orthogonality continued.

Comment on “completeness”

It can be shown that for any reasonable function *h(x)*, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where} \quad C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x').$$

Notion of completeness.

Comment on “completeness” -- continued

$$h(x) \approx \sum_n C_n f_n(x),$$

where $C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x)\right)^2$$

ϵ^2 can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2\int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x)\right) f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m}\int_a^b dx\sigma(x)h(x)f_m(x)$$

Notion of completeness and practical applications.

Variational approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle}, \qquad S(x) \equiv -\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$:

$\tilde{h}(x) = \sum_n C_n f_n(x),$ where the coefficients C_n can be assumed to be real.

A very useful property of eigenfunctions related to homework problem

Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

Therefore $$\frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

Proof of theorem continued.

Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\left\langle \tilde{h} \middle| S \middle| \tilde{h} \right\rangle}{\left\langle \tilde{h} \middle| \sigma \middle| \tilde{h} \right\rangle},$$

Example: $-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x)$ with $f_n(0) = f_n(a) = 0$

trial function $f_{\text{trial}}(x) = x(x-a)$

Exact value of $\lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$

Raleigh-Ritz estimate: $\frac{\left\langle x(a-x) \middle| -\frac{d^2}{dx^2} \middle| x(a-x) \right\rangle}{\left\langle x(a-x) \middle| x(a-x) \right\rangle} = \frac{10}{a^2}$

Example of the Rayleigh Ritz method.

A generally useful solution method -- Green's function approach

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta\left(x-x'\right)$$

Recall: Completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta\left(x-x'\right)$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\sigma\left(x\right)\sum_n\frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x')=\sum_n\frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$

The following slides present solution methods for differential equations involving the use of eigenvalues.

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_0^L G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

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From a knowledge of the Green's function we can find solutions of related inhomogeneous equations.

Example Sturm-Liouville problem:

Example: $\tau(x)=1;\ \ \sigma(x)=1;\ \ v(x)=0;\ \ a=0$ and $b=L$

$$\lambda=1; \qquad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Example.

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right)f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

In this example: $$\frac{2}{L}\sum_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right) = \delta(x-x')$$

Solution using eigenfunctions.

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Continued.

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x) = \phi_0(x) + \int_0^L G(x,x')F_0 \sin\left(\frac{\pi x'}{L}\right)dx'$$

$$= \phi_0(x) + \frac{2}{L}\sum_n \left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}\int_0^L \sin\left(\frac{n\pi x'}{L}\right)F_0 \sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$= \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1}\sin\left(\frac{\pi x}{L}\right)$$

In this case, the solution simplifies.

Alternate Green's function method:

$$G(x,x') = \frac{1}{W} g_a(x_<) g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2} - 1\right) g_i(x) = 0 \qquad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L-x);$$

$$W = g_b(x)\frac{dg_a(x)}{dx} - g_a(x)\frac{dg_b(x)}{dx} = \sin(L-x)\cos(x) + \sin(x)\cos(L-x)$$

$$= \sin(L)$$

$$\phi(x) = \phi_0(x) + \frac{\sin(L-x)}{\sin(L)} \int_0^x \sin(x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx' + \frac{\sin(x)}{\sin(L)} \int_x^L \sin(L-x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\phi(x) = \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

Another method of finding a Green's function.

General method of constructing Green's functions using homogeneous solution

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

Green's function based on homogeneous solutions (not eigenfuntions).

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right) G_\lambda(x,x') = \int_{x'-\epsilon}^{x'+\epsilon} dx\delta\left(x-x'\right)$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx \left(-\frac{d}{dx}\tau(x)\frac{d}{dx} \right) \frac{1}{W} g_a(x_<)g_b(x_>) = 1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx} g_a(x_<)g_b(x_>) \right)\Bigg]_{x'-\epsilon}^{x'+\epsilon} = \frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx}g_b(x') - g_b(x')\frac{d}{dx}g_a(x') \right)$$

$$\Rightarrow W = \tau(x')\left(g_a(x')\frac{d}{dx}g_b(x') - g_b(x')\frac{d}{dx}g_a(x') \right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'} = 0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x') = \frac{1}{W} g_a(x_<)g_b(x_>)$$

Some details.

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_{x_l}^{x_u} G_\lambda(x,x')F(x')dx'$$

$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_{x_l}^{x} g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_{x}^{x_u} g_b(x')F(x')dx'$$

More details. To be continued.