# PHY 711 Classical Mechanics and Mathematical Methods
## 10-10:50 AM MWF in Olin 103

## Discussion for Lecture 22: Chap. 7 & App. A-D (F&W)

**Generalization of the one dimensional wave equation ➔ various mathematical problems and techniques including:**

1. **Fourier transforms**
2. **Laplace transforms**
3. **Complex variables**
4. **Contour integrals**



In this lecture we will start to cover various useful mathematical techniques.

Schedule for this week

**Thurs. Oct. 21, 2021** – Yan Li, WFU graduate student – Ph. D. Defense: "First Principles Investigations of Electrolytes Materials in All-Solid-State Batteries" – 9AM-10AM **(note special time)** – mentor: Professor Natalie Holzwarth

**Thurs. Oct. 21, 2021** – Professor Jarrett Lancaster, High Point University, NC – "Simulating Quantum Dynamics with Quantum Computers" (host: D. Kim-Shapiro)

Note that Yan Li will also give a regular physics colloquium on Nov. 4th

| | | | | | |
|---|---|---|---|---|---|
| **6** | Fri, 9/03/2021 | Chap. 2 | Non-inertial coordinate systems | #5 | 9/06/2021 |
| **7** | Mon, 9/06/2021 | Chap. 3 | Calculus of Variation | #6 | 9/10/2021 |
| **8** | Wed, 9/08/2021 | Chap. 3 | Calculus of Variation | | |
| **9** | Fri, 9/10/2021 | Chap. 3 & 6 | Lagrangian Mechanics | #7 | 9/13/2021 |
| **10** | Mon, 9/13/2021 | Chap. 3 & 6 | Lagrangian Mechanics | #8 | 9/17/2021 |
| **11** | Wed, 9/15/2021 | Chap. 3 & 6 | Constants of the motion | | |
| **12** | Fri, 9/17/2021 | Chap. 3 & 6 | Hamiltonian equations of motion | #9 | 9/20/2021 |
| **13** | Mon, 9/20/2021 | Chap. 3 & 6 | Liouville theorm | #10 | 9/22/2021 |
| **14** | Wed, 9/22/2021 | Chap. 3 & 6 | Canonical transformations | | |
| **15** | Fri, 9/24/2021 | Chap. 4 | Small oscillations about equilibrium | #11 | 9/27/2021 |
| **16** | Mon, 9/27/2021 | Chap. 4 | Normal modes of vibration | #12 | 9/29/2021 |
| **17** | Wed, 9/29/2021 | Chap. 4 | Normal modes of more complicated systems | #13 | 10/04/2021 |
| **18** | Fri, 10/01/2021 | Chap. 7 | Motion of strings | #14 | 10/06/2021 |
| **19** | Mon, 10/04/2021 | Chap. 7 | Sturm-Liouville equations | | |
| **20** | Wed, 10/06/2021 | Chap.1-7 | Review | | |
| | Fri, 10/08/2021 | No class | Fall break | | |
| | Mon, 10/11/2021 | No class | Take home exam | | |
| | Wed, 10/13/2021 | No class | Take home exam | | |
| **21** | Fri, 10/15/2021 | Chap. 7 | Sturm-Liouville equations -- exam due | | |
| **22** | Mon, 10/18/2021 | Chap. 7 | Fourier and other transform methods | #15 | 10/20/2021 |
| **23** | Wed, 10/20/2021 | Chap. 7 | Complex variables and contour integration | | |

This is the schedule. You will receive an email containing the mid term exam. It will be due next Monday.

This assignment covers material from Friday's lecture --

# PHY 711 -- Assignment #15

Oct. 18, 2021

Continue reading Chapter 7 in **Fetter & Walecka**.

> Consider the example presented in Lecture 21, slide 23, where a one-dimensional Poisson equation was solved using a Green's function constructed from the corresponding homogenious solutions. Verify the results on this slide and check that the resultant potential $\Phi(x)$ satisfies the particular Poisson equation for $x \leq -a$, $-a \leq x \leq a$, and for $x \geq a$.

## Review – Sturm-Liouville equations defined over a range of x.

For $x_a \le x \le x_b$

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.



Review of the Sturm Liouville equations.

Formal statement of the completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta\left(x-x'\right) \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx\sigma(x)\left(f_n(x)\right)^2$$

Example for $\tau(x)=1=\sigma(x)$ and $v(x)=0$ with

$0 \leq x \leq L$ and $f_n(0)=0=f_n(L)$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x) \quad \Rightarrow -\frac{d^2 f_n(x)}{dx^2}=\lambda_n f_n(x$$

In this case, the normalized eigenfunctions are

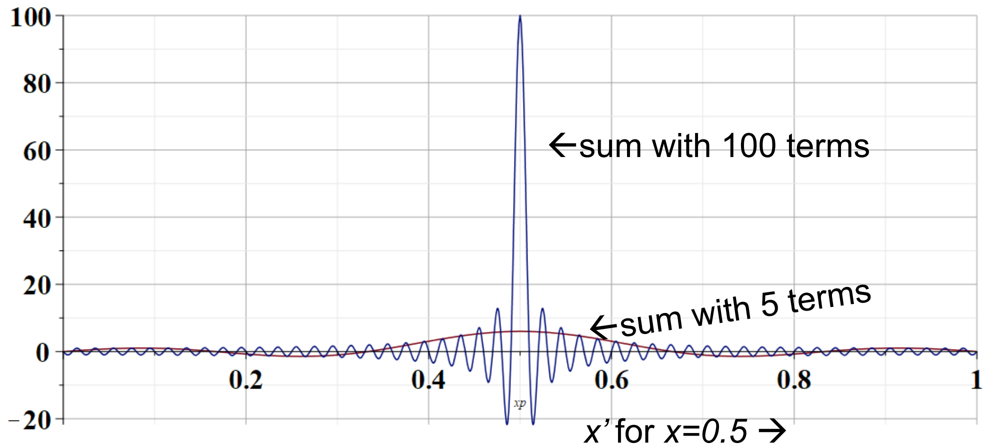
$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n=\left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,.....$$



Specializing to the simplest case.

Formal completeness for this case:

$$\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta\left(x-x'\right)\quad \text{for}\quad 0\leq x\leq L$$

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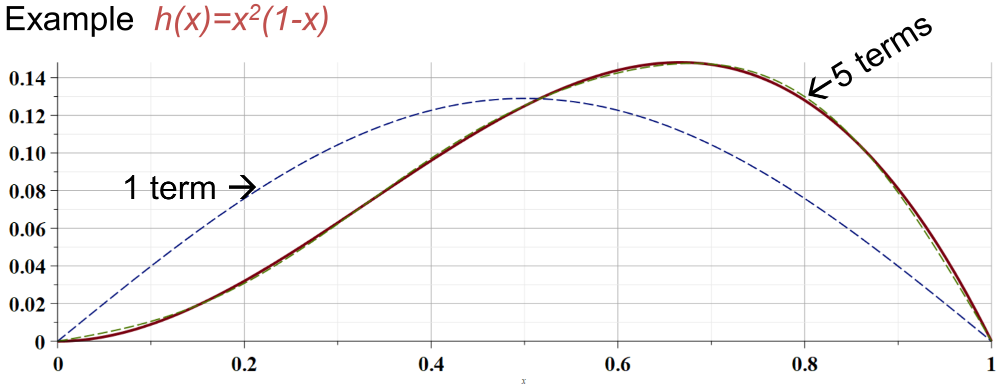
Visualizing the completeness condition.

**Joseph Fourier**


Jean-Baptiste Joseph Fourier

| | |
|---|---|
| **Born** | 21 March 1768<br>Auxerre, Burgundy, Kingdom of France (now in Yonne, France) |
| **Died** | 16 May 1830 (aged 62) |

Example *h(x)=x²(1-x)*




$$\tilde{h}(x,N) = \sum_{n=1}^{N} A_n \sin(n\pi x)$$

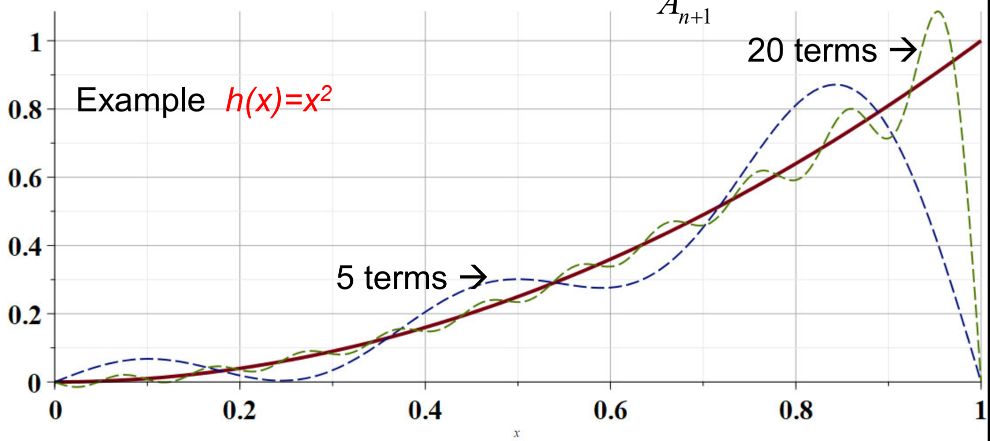
$$A_n = \int_0^1 2\sin(n\pi x')h(x')dx' = -\frac{4\left(1+2(-1)^n\right)}{n^3\pi^3}$$



Numerical evaluation of an example.

## Convergence of the Fourier series

In general, $\tilde{h}(x, N \rightarrow \infty) \approx h(x)$ if $\dfrac{A_n}{A_{n+1}} < 1$




$$\tilde{h}(x, N) = \sum_{n=1}^{N} A_n \sin(n\pi x) \qquad A_n == -\frac{2\left(2 + (-1)^n (n^2\pi^2 - 2)\right)}{n^3\pi^3}$$



Numerical evaluation of less convergent example.

## Using Fourier series to solve the wave equation.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

In this case, we will impose the boundary values $u(0,t) = 0 = u(L,t)$, and the initial conditions $u(x,0) = \varphi(x)$ and $\frac{\partial u(x,0)}{\partial t} = \psi(x)$.

Now suppose that $u(x,t) = \rho(x)\cos(\omega t + \alpha)$ where $\omega$ and $\alpha$ are not yet known. The spatial function $\rho(x)$ must then satisfy

$$-\frac{d^2\rho(x)}{dx^2} = \frac{\omega^2}{c^2}\rho(x) \equiv k^2\rho(x) \quad \text{with } \rho(0) = \rho(L) = 0$$

We recognize this equation and find the normalized eigenfunctions to be

$$\rho_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad k_n^2 = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,\ldots.. \qquad \omega_n = k_n c$$



Using Fourier methods to solve the wave equation.

Using Fourier series to solve the wave equation -- continued. The general solution can be formed by taking a linear combination of the eigenfunction results.

$$u(x,t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \alpha_n)$$

where $\rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, ..... \quad \omega_n = \frac{n\pi}{L} c$

The constants $C_n$ and $\alpha_n$ are determined from the initial conditions.

$$\tilde{\varphi}(x) = \sum_{n=1}^{\infty} \varphi_n \rho_n(x) \quad \text{where } \varphi_n \equiv \int_0^L \rho_n(x') \varphi(x') dx'$$

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \psi_n \rho_n(x) \quad \text{where } \psi_n \equiv \int_0^L \rho_n(x') \psi(x') dx'$$



Setting the boundary and initial conditions.

Using Fourier series to solve the wave equation -- continued. Finding the constants from the eigenfunction (Fourier) expansion.

$$u(x,t)=\sum_{n=1}^{\infty} C_n \rho_n(x)\cos(\omega_n t+\alpha_n)$$

$$u(x,0)=\sum_{n=1}^{\infty} C_n \cos(\alpha_n)\rho_n(x)=\sum_{n=1}^{\infty}\varphi_n\rho_n(x)$$

$$\frac{\partial u(x,0)}{\partial t}=-\sum_{n=1}^{\infty}\omega_n C_n \sin(\alpha_n)\rho_n(x)=\sum_{n=1}^{\infty}\psi_n\rho_n(x)$$

Since the eigenfunctions $\rho_n(x)$ are orthogonal, the constants are immediately determined:

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\cos(\alpha_n)\cos(\omega_n t)-C_n\sin(\alpha_n)\sin(\omega_n t)\right)\rho_n(x)$$

$$=\sum_{n=1}^{\infty}\left(\varphi_n\cos(\omega_n t)+\frac{\psi_n}{\omega_n}\sin(\omega_n t)\right)\rho_n(x)$$



Determining the constants.

Solution to wave equation from eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} \left( \varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \rho_n(x)$$

where $\rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ $\qquad \omega_n = \frac{n\pi}{L} c$

Recall D'Alembert's solution

$$u(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x')dx'$$

Are these two solutions

a. Identical
b. Equivalent
c. Totally different



Interesting question.

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

In the next several slides we will consider a related concept – the Laplace transform.



We now consider another technique that is uses to solve initial value equations.

## Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty \mathrm{e}^{-px}\phi(x)dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$



A quick introduction to Laplace transform methods.

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by $e^{-px}$ and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$



An example.

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - e^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \phi(0) = 0 \qquad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$



Some details.

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0)=0,\ \frac{d\phi(0)}{dx}=0$$

$$\mathcal{L}_\phi(p)=-\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2+\left(\frac{\pi}{L}\right)^2\right)}$$

$$=-F_0\left(\frac{\pi/L}{(\pi/L)^2-1}\right)\left(\frac{1}{p^2+1}-\frac{1}{p^2+\left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at)e^{-pt}dt=\frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x)=\frac{F_0}{(\pi/L)^2-1}\left(\sin\left(\frac{\pi x}{L}\right)-\frac{\pi}{L}\sin(x)\right)$$

Does this result look familiar?

a. Yes
b. No



More details.

Table of Laplace transforms

## Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

| $F(s)$ | $f(t) \quad 0 \le t$ | |
|---|---|---|
| 1. $1$ | $\delta(t)$ | unit impulse at $t=0$ |
| 2. $\frac{1}{s}$ | 1 or $u(t)$ | unit step starting at $t=0$ |
| 3. $\frac{1}{s^2}$ | $t \cdot u(t)$ or $t$ | ramp function |
| 4. $\frac{1}{s^n}$ | $\frac{1}{(n-1)!}t^{n-1}$ | n = positive integer |
| 5. $\frac{1}{s}e^{-as}$ | $u(t-a)$ | unit step starting at $t=a$ |
| 6. $\frac{1}{s}(1-e^{-as})$ | $u(t)-u(t-a)$ | rectangular pulse |
| 7. $\frac{1}{s+a}$ | $e^{-at}$ | exponential decay |
| 8. $\frac{1}{(s+a)^n}$ | $\frac{1}{(n-1)!}t^{n-1}e^{-at}$ | n = positive integer |
| 9. $\frac{1}{s(s+a)}$ | $\frac{1}{a}(1-e^{-at})$ | |
| 10. $\frac{1}{s(s+a)(s+b)}$ | $\frac{1}{ab}(1-\frac{b}{b-a}e^{-at}+\frac{a}{b-a}e^{-bt})$ | |
| 11. $\frac{s+\alpha}{s(s+a)(s+b)}$ | $\frac{1}{ab}[\alpha-\frac{b(\alpha-a)}{b-a}e^{-at}+\frac{a(\alpha-b)}{b-a}e^{-bt}]$ | |
| 12. $\frac{1}{(s+a)(s+b)}$ | $\frac{1}{b-a}(e^{-at}-e^{-bt})$ | |
| 13. $\frac{s}{(s+a)(s+b)}$ | $\frac{1}{a-b}(ae^{-at}-be^{-bt})$ | |

https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf



Table of transforms for simple functions.

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t)dt$$

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dp$$

In order to evaluate these integrals, we need to use complex analysis.

Check: $$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\varphi(p)dp = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^\infty e^{-pu}\varphi(u)du$$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp = \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\left(e^{\lambda(t-u)}2\pi i\ \delta(t-u)\right)$$

$$= \begin{cases} \varphi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Mathematical treatment of general case.

## Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

## Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

## Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$



Introduction to properties of complex numbers.

## Analytic function

$f(z)$ is analytic if it is:

- ∘ continuous
- ∘ single valued
- ∘ its first derivative satisfies Cauchy-Rieman conditions

Which of the following functions are analytic?

$$f(z) = e^{z}$$

$$f(z) = z^{n}$$

$$f(z) = \ln z$$

$$f(z) = z^{\alpha}$$



Notion of analytic function. Some of these functions are not analytic.

## Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = \left(x + iy\right)^2 = \left(x^2 - y^2\right) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$



Some details. To be continued.