PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 22: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. **Fourier transforms**
2. **Laplace transforms**
3. **Complex variables**
4. **Contour integrals**

In this lecture we will start to cover various useful mathematical techniques.

Schedule for this week

Thurs. Oct. 21, 2021 – Yan Li, WFU graduate student – Ph. D. Defense: "First Principles Investigations of Electrolytes Materials in All-Solid-State Batteries" – 9AM-10AM **(note special time)** – mentor: Professor Natalie Holzwarth

Thurs. Oct. 21, 2021 – Professor Jarrett Lancaster, High Point University, NC – "Simulating Quantum Dynamics with Quantum Computers" (host: D. Kim-Shapiro)

Note that Yan Li will also give a regular physics colloquium on Nov. 4th

Your questions –
From Owen -- Does the number of terms in the summation for the Fourier series to converge depend on the type of function being analyzed?

6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021
10	Mon, 9/13/2021	Chap. 3 & 6	Lagrangian Mechanics	#8	9/17/2021
11	Wed, 9/15/2021	Chap. 3 & 6	Constants of the motion		
12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/20/2021
13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorm	#10	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/20/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration		

This is the schedule. You will receive an email containing the mid term exam. It will be due next Monday.

This assignment covers material from Friday's lecture --

PHY 711 -- Assignment #15

Oct. 18, 2021

Continue reading Chapter 7 in **Fetter & Walecka**.

> Consider the example presented in Lecture 21, slide 23, where a one-dimensional Poisson equation was solved using a Green's function constructed from the corresponding homogenious solutions. Verify the results on this slide and check that the resultant potential $\Phi(x)$ satisfies the particular Poisson equation for $x \leq -a$, $-a \leq x \leq a$, and for $x \geq a$.

Review – Sturm-Liouville equations defined over a range of x.

For $x_a \le x \le x_b$

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.

Review of the Sturm Liouville equations.

Formal statement of the completeness of eigenfunctions:

$$\sigma\left(x\right)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta\left(x - x'\right) \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx\sigma(x)\left(f_n(x)\right)^2$$

Example for $\tau(x) = 1 = \sigma(x)$ and $v(x) = 0$ with

$0 \le x \le L$ and $f_n(0) = 0 = f_n(L)$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right)f_n(x) = \lambda_n\sigma(x)f_n(x) \quad \Rightarrow -\frac{d^2 f_n(x)}{dx^2} = \lambda_n f_n(x$$

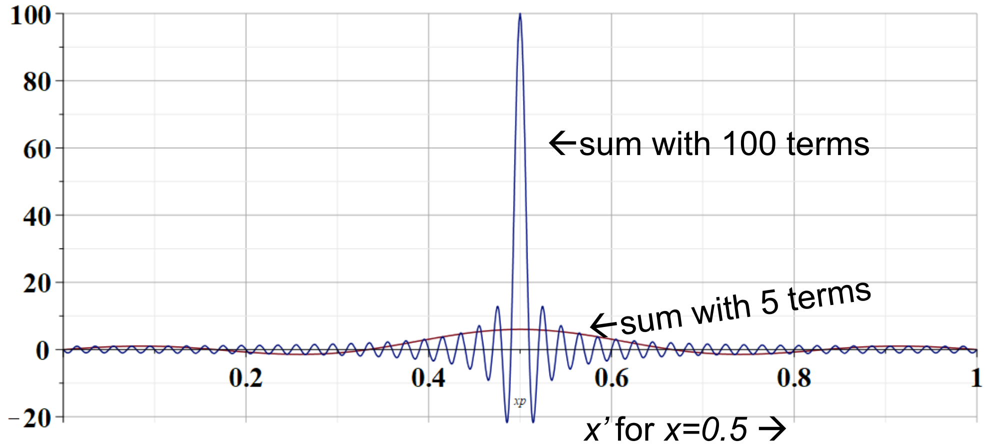
In this case, the normalized eigenfunctions are

$$f_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, ...$$

Specializing to the simplest case.

Formal completeness for this case:

$$\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta\left(x-x'\right) \quad \text{for} \quad 0\le x\le L$$



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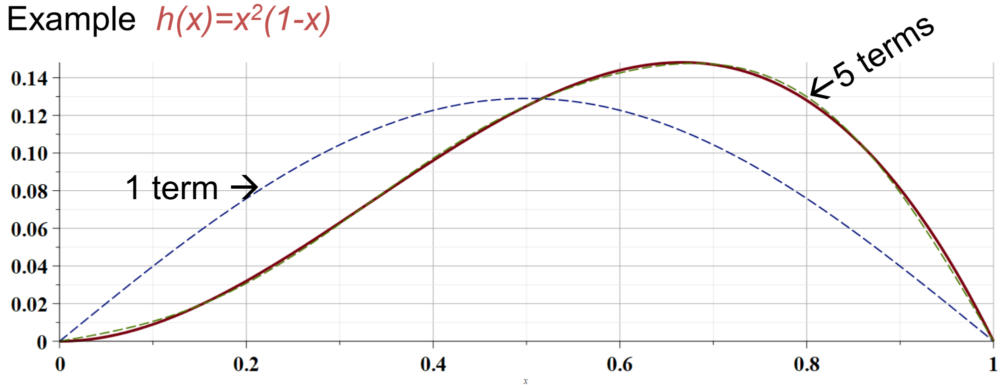
Visualizing the completeness condition.

Joseph Fourier

Jean-Baptiste Joseph Fourier

Born	21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France)
Died	16 May 1830 (aged 62)

Example $h(x)=x^2(1-x)$



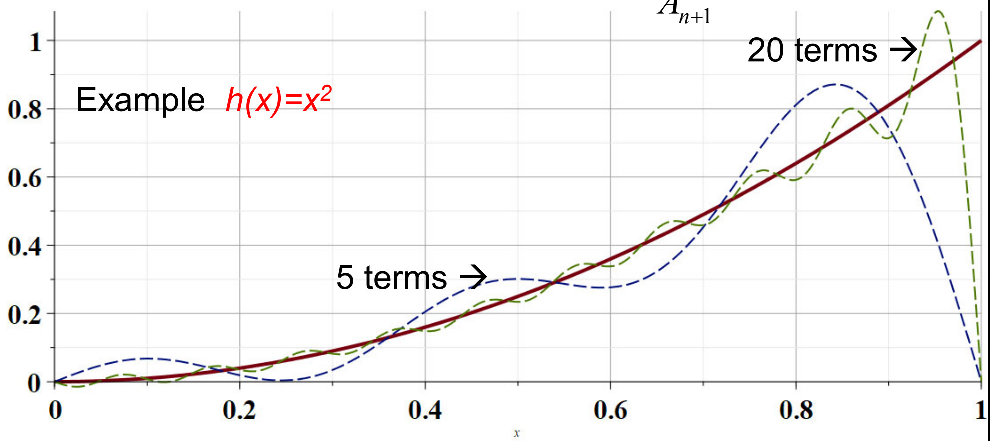
$$\tilde{h}(x,N)=\sum_{n=1}^{N} A_n \sin(n\pi x)$$

$$A_n=\int_0^1 2\sin(n\pi x')h(x')dx'=-\frac{4\left(1+2(-1)^n\right)}{n^3\pi^3}$$

Numerical evaluation of an example.

Convergence of the Fourier series

In general, $\tilde{h}(x, N \to \infty) \approx h(x)$ if $\dfrac{A_n}{A_{n+1}} < 1$



$$\tilde{h}(x,N) = \sum_{n=1}^{N} A_n \sin(n\pi x) \qquad A_n == -\frac{2\left(2+(-1)^n (n^2\pi^2 - 2)\right)}{n^3\pi^3}$$

Numerical evaluation of less convergent example.

From your question -- Does the number of terms in the summation for the Fourier series to converge depend on the type of function being analyzed?

For $\quad h(x) = x^2 = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$

Approximation with a finite summation:

$$\tilde{h}(x, N) = \sum_{n=1}^{N} A_n \sin(n\pi x) \quad \text{where } A_n = -\frac{2\left(2 + (-1)^n (n^2\pi^2 - 2)\right)}{n^3\pi^3}$$

Using Fourier series to solve the wave equation.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

In this case, we will impose the boundary values $u(0,t) = 0 = u(L,t),$ and

the initial conditions $u(x,0) = \varphi(x)$ and $\frac{\partial u(x,0)}{\partial t} = \psi(x).$

Now suppose that $u(x,t) = \rho(x)\cos(\omega t + \alpha)$ where ω and α are not yet known.

The spatial function $\rho(x)$ must then satisfy

$$-\frac{d^2\rho(x)}{dx^2} = \frac{\omega^2}{c^2}\rho(x) \equiv k^2\rho(x) \quad \text{with } \rho(0) = \rho(L) = 0$$

We recognize this equation and find the normalized eigenfunctions to be

$$\rho_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad k_n^2 = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, ... \qquad \omega_n = k_n c$$

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Using Fourier methods to solve the wave equation.

Using Fourier series to solve the wave equation -- continued.
The general solution can be formed by taking a linear combination of the eigenfunction results.

$$u(x,t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \alpha_n)$$

where $\rho_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, ... \quad \omega_n = \frac{n\pi}{L} c$

The constants C_n and α_n are determined from the initial conditions.

$$\tilde{\varphi}(x) = \sum_{n=1}^{\infty} \varphi_n \rho_n(x) \quad \text{where } \varphi_n \equiv \int_0^L \rho_n(x') \varphi(x') dx'$$

$$\tilde{\psi}(x) = \sum_{n=1}^{\infty} \psi_n \rho_n(x) \quad \text{where } \psi_n \equiv \int_0^L \rho_n(x') \psi(x') dx'$$

Setting the boundary and initial conditions.

Using Fourier series to solve the wave equation -- continued. Finding the constants from the eigenfunction (Fourier) expansion.

$$u(x,t)=\sum_{n=1}^{\infty}C_n\rho_n(x)\cos(\omega_n t+\alpha_n)$$

$$u(x,0)=\sum_{n=1}^{\infty}C_n\cos(\alpha_n)\rho_n(x)=\sum_{n=1}^{\infty}\varphi_n\rho_n(x)$$

$$\frac{\partial u(x,0)}{\partial t}=-\sum_{n=1}^{\infty}\omega_n C_n\sin(\alpha_n)\rho_n(x)=\sum_{n=1}^{\infty}\psi_n\rho_n(x)$$

Since the eigenfunctions $\rho_n(x)$ are orthogonal, the constants are immediately determined:

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\cos(\alpha_n)\cos(\omega_n t)-C_n\sin(\alpha_n)\sin(\omega_n t)\right)\rho_n(x)$$

$$=\sum_{n=1}^{\infty}\left(\varphi_n\cos(\omega_n t)+\frac{\psi_n}{\omega_n}\sin(\omega_n t)\right)\rho_n(x)$$

Determining the constants.

Solution to wave equation from eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty}\left(\varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n}\sin(\omega_n t) \right)\rho_n(x)$$

where $\rho_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad \omega_n = \frac{n\pi}{L}c$

Recall D'Alembert's solution

$$u(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x')dx'$$

Are these two solutions

a. Identical
b. Equivalent
c. Totally different

Interesting question.

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

In the next several slides we will consider a related concept – the Laplace transform.

We now consider another technique that is uses to solve initial value equations.

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty \mathrm{e}^{-px}\phi(x)dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$

A quick introduction to Laplace transform methods.

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

An example.

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty \mathrm{e}^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x\mathrm{e}^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{\mathrm{e}^{-\gamma x}} = \int_0^\infty \mathrm{e}^{-\gamma x}\mathrm{e}^{-px} dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - \mathrm{e}^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \phi(0) = 0 \qquad \text{and} \qquad \frac{d\phi}{dx}(0) = 0$$

Some details.

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \qquad \text{with} \quad \phi(0)=0,\ \frac{d\phi(0)}{dx}=0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2+\left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0\left(\frac{\pi/L}{(\pi/L)^2-1}\right)\left(\frac{1}{p^2+1}-\frac{1}{p^2+\left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at)e^{-pt}dt = \frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2-1}\left(\sin\left(\frac{\pi x}{L}\right)-\frac{\pi}{L}\sin(x)\right)$$

Does this result look familiar?

a. Yes
b. No

More details.

Table of Laplace transforms

Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

	$F(s)$	$f(t) \quad 0 \le t$	
1.	1	$\delta(t)$	unit impulse at $t = 0$
2.	$\frac{1}{s}$	1 or $u(t)$	unit step starting at $t = 0$
3.	$\frac{1}{s^2}$	$t \cdot u(t)$ or t	ramp function
4.	$\frac{1}{s^n}$	$\frac{1}{(n-1)!} t^{n-1}$	n = positive integer
5.	$\frac{1}{s} e^{-as}$	$u(t-a)$	unit step starting at $t = a$
6.	$\frac{1}{s}(1 - e^{-as})$	$u(t) - u(t-a)$	rectangular pulse
7.	$\frac{1}{s+a}$	e^{-at}	exponential decay
8.	$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$	n = positive integer
9.	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1 - e^{-at})$	
10.	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}(1 - \frac{b}{b-a} e^{-at} + \frac{a}{b-a} e^{-bt})$	
11.	$\frac{s+\alpha}{s(s+a)(s+b)}$	$\frac{1}{ab}[\alpha - \frac{b(\alpha-a)}{b-a} e^{-at} + \frac{a(\alpha-b)}{b-a} e^{-bt}]$	
12.	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	
13.	$\frac{s}{(s+a)(s+b)}$	$\frac{1}{a-b}(ae^{-at} - be^{-bt})$	

https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf

Table of transforms for simple functions.

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t)dt$$

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dp$$

In order to evaluate these integrals, we need to use complex analysis.

Check: $\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\varphi(p)dp = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^\infty e^{-pu}\varphi(u)du$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp = \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\left(e^{\lambda(t-u)}2\pi i\ \delta(t-u)\right)$$

$$= \begin{cases} \varphi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Mathematical treatment of general case.

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

Introduction to properties of complex numbers.

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Which of the following functions are analytic?

$$f(z) = e^z$$

$$f(z) = z^n$$

$$f(z) = \ln z$$

$$f(z) = z^\alpha$$

Notion of analytic function. Some of these functions are not analytic.

Some details

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x \sin(y) = -\frac{\partial u}{\partial y}$$

$$z^2 = \left(x+iy\right)^2 = \left(x^2 - y^2\right) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

Some details. To be continued.