PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 23: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation ➔ various mathematical problems and techniques including:

1. **Fourier transforms**
2. **Laplace transforms**
3. **Complex variables**
4. **Contour integrals**

In this lecture, we will focus on the mathematics of complex variables.

12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/20/2021
13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorm	#10	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021

No new homework while working on mid term exam.

10/20/2021

PHY 711 – Homework # 16

Read Appendix A of **Fetter and Walecka**.

1. Assume that $a > 0$ and $b > 0$; use contour integration methods to evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2} dx.$$

Note that you may use Maple, Mathematica, or other software to evaluate this integral, but full credit will be earned by using the contour integration methods.

Schedule for Thursday 10/21/2021 --

Thurs. Oct. 21, 2021 – Yan Li, WFU graduate student – Ph. D. Defense: "First Principles Investigations of Electrolytes Materials in All-Solid-State Batteries" – 9AM-10AM **(note special time)** – mentor: Professor Natalie Holzwarth

ZSR auditorium and via zoom
(reception following defense ~ 11:15 AM in Olin Lobby)

Thurs. Oct. 21, 2021 – Professor Jarrett Lancaster, High Point University, NC – "Simulating Quantum Dynamics with Quantum Computers" (host: D. Kim-Shapiro)

4 PM in Olin 101 and via zoom
(reception prior to colloquium at 3:30 PM in Olin Lobby)

Introduction to complex variables

1. Basic properties
2. Notion of an analytic complex function
3. Cauchy integral theory
4. Analytic functions and functions with poles
5. Evaluating integrals of functions in the complex plane

We will review/introduce the basic ideas associated with complex variables.

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\ e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

First we consider the basic definitions and representations of a complex number. Then we consider a function of complex numbers. The Cauchy relationships follow from the notion that a function that is differentiable in the complex plane must have consistent partial derivatives along the real and imaginary axes.

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Examples of analytic functions

$$e^z = e^{x+iy} = e^x\cos(y) + ie^x\sin(y)$$

$$\frac{\partial u}{\partial x} = e^x\cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^x\sin(y) = -\frac{\partial u}{\partial y} \quad \checkmark$$

$$z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} \quad \checkmark$$

Notion of an analytic function and an example that satisfies the conditions.

Examples of non-analytic functions

Note that $z = \rho e^{i\phi} = \rho e^{i\phi + i2\pi n}$ for any integer n

$$\Rightarrow \ln z = \ln \rho + i\left(\phi + 2\pi n\right)$$

$\ln z$ is not analytic because it is multivalued

$$\Rightarrow z^{\alpha} = \rho^{\alpha} e^{i\alpha\phi} e^{i2\pi n\alpha}$$

z^{α} is not analytic for non-integer α because it is multivalued

Behavior of $f(z) = \dfrac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Examples of non-analytic functions. Special property of contour integrals about a function with a simple "pole".

Behavior of $f(z)=\frac{1}{z^n}$ about the point $z=0$:

For an integer n, consider

$$\oint \frac{1}{z^n}dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n}\int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

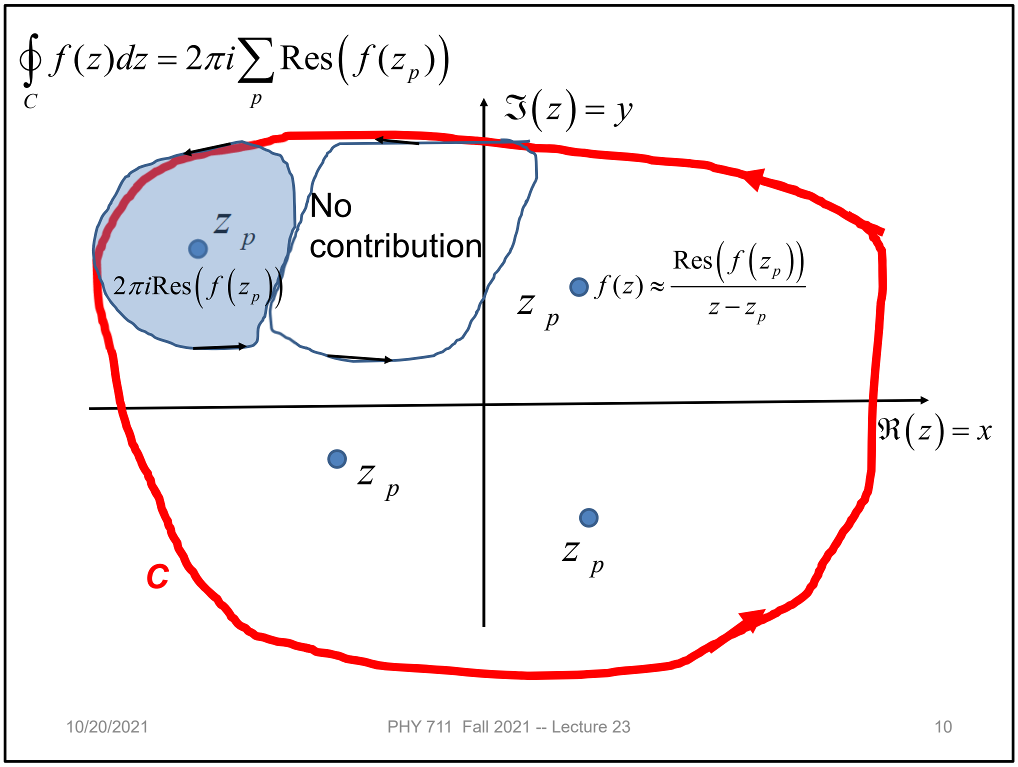
This observation helps us to focus on a special kind of singularity called a "pole"

For $f(z)$ in the vicinity of $z=z_p$: $f(z) \approx \frac{g(z_p)}{z-z_p}$

Therefore: $\oint f(z)dz = 0$ or $\oint f(z)dz = g(z_p)\oint \frac{dz}{z-z_p} = 2\pi i g(z_p)$

Integration does not include z_p

Integration does include z_p



Contributions to a closed contour from various contributions.

General formula for determining residue:

Suppose that in the neighborhood of z_p, $f(z) \approx \frac{h(z)}{\left(z-z_p\right)^m} \underset{z\to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z-z_p}$

Since $h(z) \equiv \left(z-z_p\right)^m f(z)$ is analytic near z_p, we can make a Taylor exansion

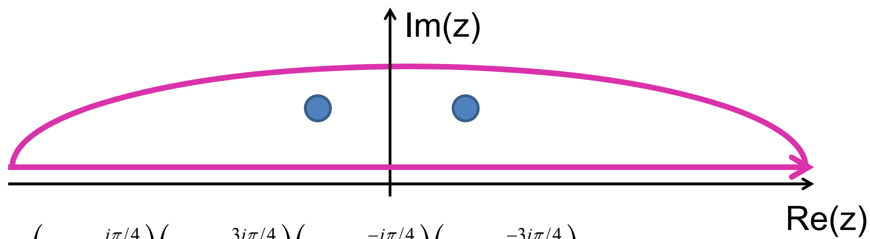
about z_p : $\quad h(z) \approx h(z_p) + \left(z-z_p\right)\frac{dh(z_p)}{dz} + \ldots + \frac{\left(z-z_p\right)^{m-1}}{(m-1)!}\frac{d^{m-1}h(z_p)}{dz^{m-1}} + \ldots.$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z\to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}}\right\}$$

In the following examples $m=1$

Residue theorem

Example: $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx + 0 = \oint \frac{z^2}{1+z^4} dz$



$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint \frac{z^2}{1+z^4} dz = 2\pi i \left(\mathrm{Res}\left(z_p = e^{i\pi/4}\right) + \mathrm{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

Note: *m=1*

$$\mathrm{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \mathrm{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4} dz = 2\pi i \left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{2}\left(\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) - \left(-\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Example of a contour integral using the residue theorem.

Some details:

Note that: $e^{i\pi} = -1 = e^{-i\pi}$

$$e^{-3i\pi/4} = e^{i\pi/4 - i\pi} = -e^{i\pi/4}$$

$$f(z) = \frac{z^2}{1+z^4}$$

$$\text{Res}\left(f(z = e^{i\pi/4}\right) = \frac{\left(e^{i\pi/4}\right)^2}{\left(e^{i\pi/4} - e^{3i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-3i\pi/4}\right)}$$

$$= \frac{e^{i\pi/2}}{\left(e^{i\pi/4} + e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} + e^{i\pi/4}\right)}$$

$$= \frac{e^{i\pi/4}}{2\left(i-(-i)\right)} = \frac{e^{i\pi/4}}{4i}$$

Question – Could we have chosen the contour in the lower half plane?

a. Yes   b. No

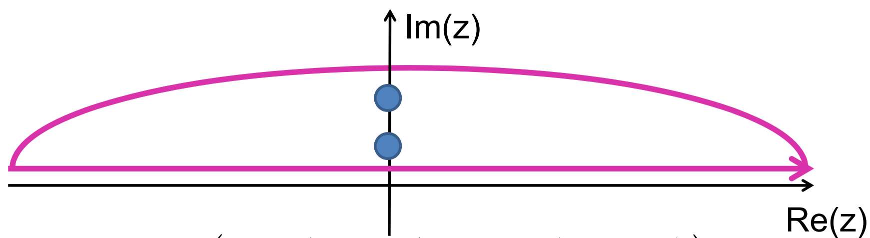
Some details.

Another example: $I = \int_0^\infty \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx$.

$$\int_0^\infty \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4 + 5x^2 + 1} dx = \frac{1}{2} \oint \frac{e^{iaz}}{4z^4 + 5z^2 + 1} dz$$

$$4z^4 + 5z^2 + 1 = 4(z - i)\left(z - \tfrac{i}{2}\right)(z + i)\left(z + \tfrac{i}{2}\right)$$

Note: *m=1*



$$I = 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

Another example.

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$=2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$=\frac{\pi}{6}\left(-e^{-a} + 2e^{-a/2}\right)$$

Question – Could we have chosen the contour in the lower half plane?

a. Yes b. No

Note that for $a > 0$ and $z_I > 0$

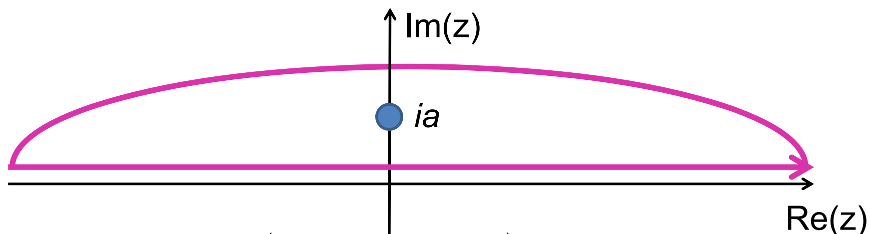
in the lower half plane: $e^{iaz} = e^{iaz_R}e^{az_I}$

Some details.

Another example: $I = \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx \quad \text{for } k > 0 \text{ and } a > 0$

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$



$$I = 2\pi i \left(\text{Res}\left(z_p = ia \right) \right) = \pi e^{-ka}$$

Another example

Some details --

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2+a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{xe^{ikx}}{x^2+a^2} dx = \frac{1}{i} \oint \frac{ze^{ikz}}{z^2+a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$

$$\frac{1}{i} \oint \frac{ze^{ikz}}{z^2+a^2} dz = 2\pi i \frac{1}{i} \lim_{z \to ia} \left((z - ia) \frac{ze^{ikz}}{z^2+a^2} \right)$$

$$= 2\pi i \frac{1}{i} \frac{iae^{-ka}}{2ia} = \pi e^{-ka}$$

More details.

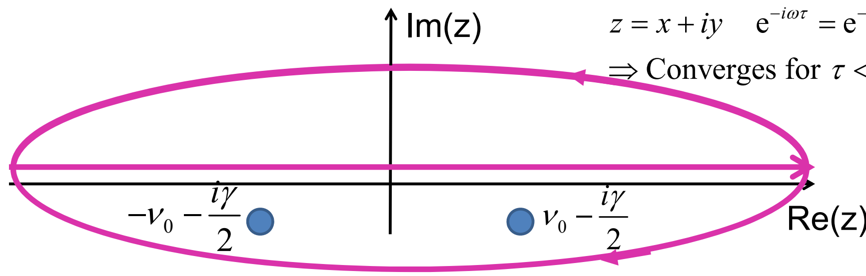
From the Drude model of dielectric response --

$$G(\tau)=\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}$$ where ω_p, ω_0, and γ are positive constants

Upper hemisphere:

$z = x + iy \quad \mathrm{e}^{-i\omega\tau} = \mathrm{e}^{-ix\tau + y\tau}$

$\Rightarrow$ Converges for $\tau < 0$



$$\nu_0 \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

Lower hemisphere:

$z = x - iy \quad \mathrm{e}^{-i\omega\tau} = \mathrm{e}^{-ix\tau - y\tau}$

$\Rightarrow$ Converges for $\tau > 0$

From the Drude model of dielectric response -- continued --

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$ where ω_p, ω_0, and γ are positive constants

$$G(\tau) = \omega_p^2 \begin{cases} 0 & \text{for } \tau < 0 \\ e^{-\gamma\tau/2} \dfrac{\sin \nu_0 \tau}{\nu_0} & \text{for } \tau > 0 \end{cases}$$

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Another example from the Drude model.

Cauchy integral theorem for analytic function *f(z)*:

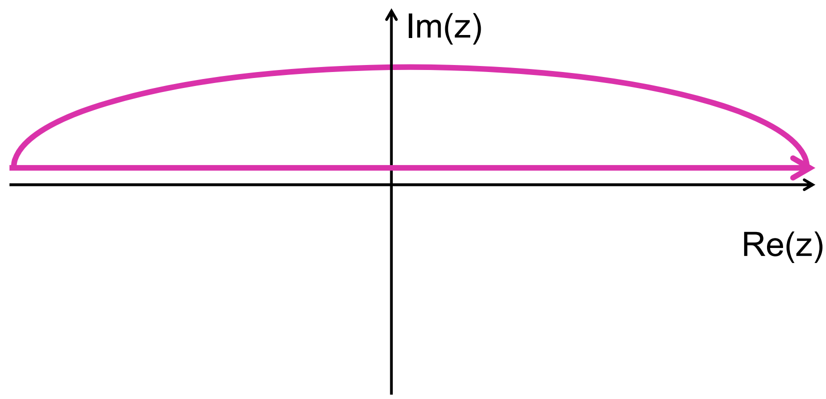
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'.$$

Another useful theorem from Cauchy.

Example

$$\text{Suppose } f(|z| \rightarrow \infty) = 0 \text{ and for } z = x:$$

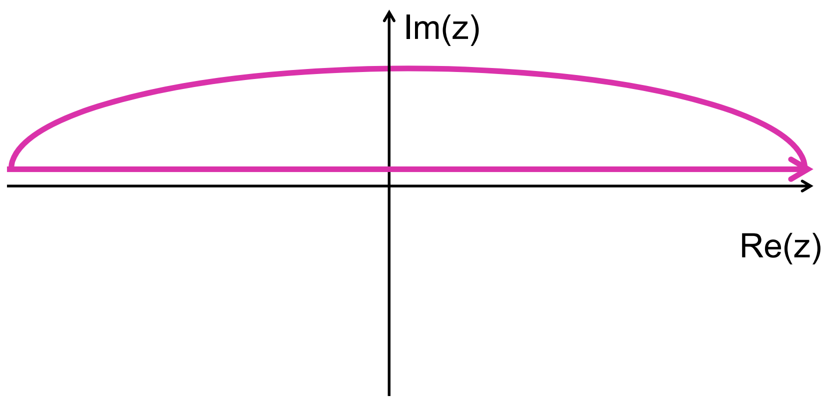
$$f(x) = a(x) + ib(x)$$



Considering real and imaginary parts.

Example -- continued

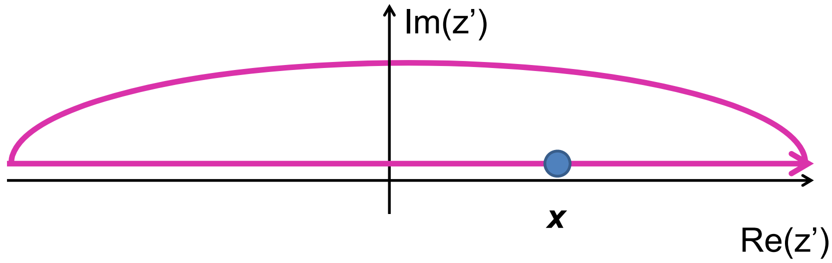
$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-z}dz' \quad \text{where} \quad f(x) = a(x) + ib(x)$$



$$a(x) + ib(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{a(x') + ib(x')}{x' - x}dx' + 0$$

Real and imaginary parts for Cauchy’s integral relation.

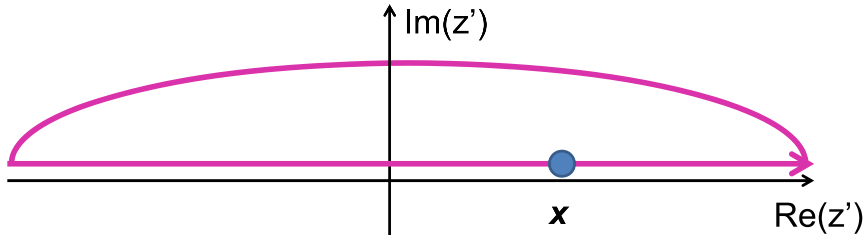
Example -- continued



$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' = \int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx' + \int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx' + \int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

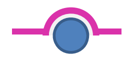
$$= P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx' + i\pi\, f(x)$$

Detail of how to evaluate the point x=x' in terms of the principal parts integral.



let $u = x' - x$

let $x \rightarrow x + i\eta$



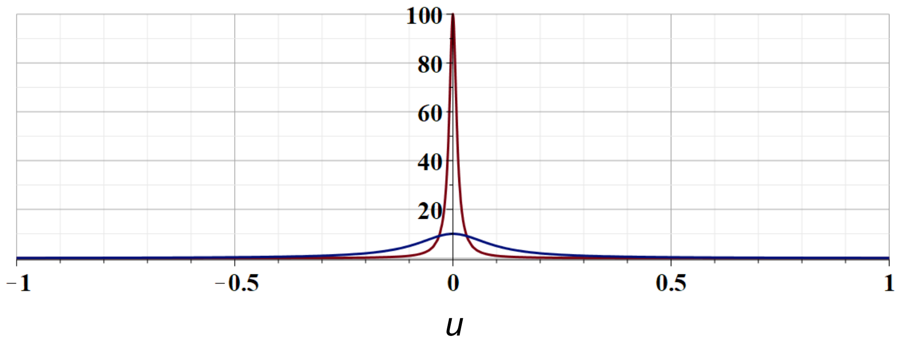
$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x} dx' \approx f(x) \lim_{\eta\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{1}{u - i\eta} du = f(x) \lim_{\eta\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{u + i\eta}{u^2 + \eta^2} du$$

$$= i\pi f(x) \qquad \text{since} \qquad \lim_{\eta\to 0} \frac{i\eta}{u^2+\eta^2} \approx i\pi\delta(u)$$

Some details.

More details --

$$\lim_{\eta \to 0} \frac{\eta}{u^2 + \eta^2} \approx \pi \delta(u)$$



Justification of the delta function result.

Example -- continued

$$\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' = \int_{-\infty}^{x-\varepsilon} \frac{f(x')}{x'-x} dx' + \int_{x+\varepsilon}^{\infty} \frac{f(x')}{x'-x} dx' + \int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x} dx'$$

$$= P\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' + i\pi\ f(x)$$

$$a(x) + ib(x) = \frac{P}{2\pi i} \int_{-\infty}^{\infty} \frac{a(x') + ib(x')}{x'-x} dx' + \frac{\pi i}{2\pi i}(a(x) + ib(x))$$

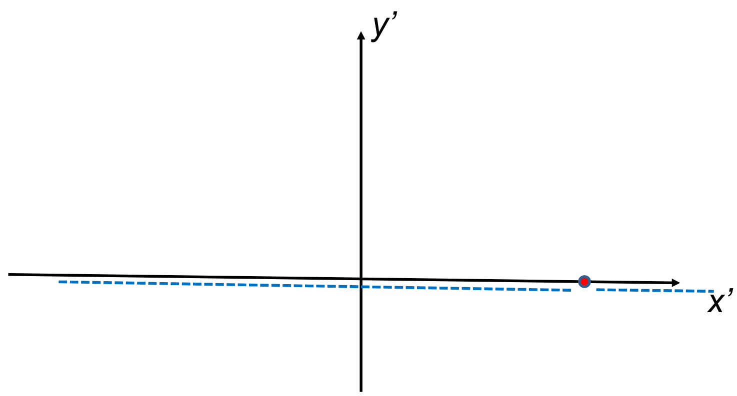
$$\Rightarrow a(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{b(x')}{x'-x} dx' \qquad b(x) = -\frac{P}{\pi} \int_{-\infty}^{\infty} \frac{a(x')}{x'-x} dx'$$

Kramers-Kronig relationships

Final result relating real and imaginary parts of complex function.

Comment on evaluating principal parts integrals

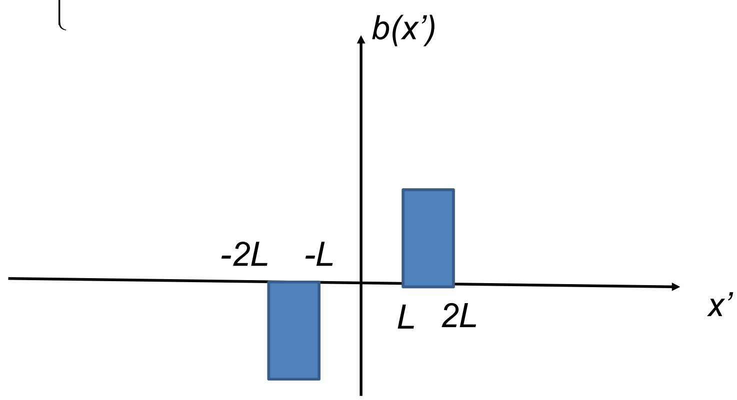
$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x}dx' \right)$$



Some details.

Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \quad L < x' < 2L \\ -B_0 & \text{for } \quad -2L < x' < -L \end{cases}$$



Simple example for complex function.

Example:

$$b(x')=\begin{cases} 0 & \text{for } x'<-2L, \quad -L<x'<L, \quad x'>2L \\ B_0 & \text{for } \quad L<x'<2L \\ -B_0 & \text{for } \quad -2L<x'<-L \end{cases}$$

$$a(x)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \lim_{\epsilon\to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon}\frac{b(x')}{x'-x}dx'+\frac{1}{\pi}\int_{x+\epsilon}^{\infty}\frac{b(x')}{x'-x}dx'\right)$$

For $x<-2L \quad \text{or} \quad x>2L \quad -L<x<L$:

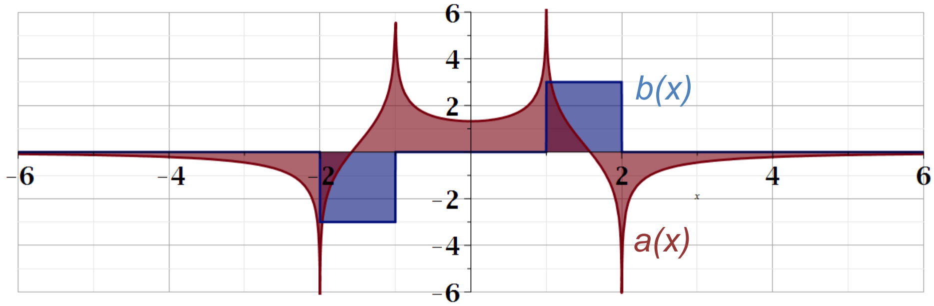
$$a(x)=\frac{-B_0}{\pi}\int_{-2L}^{-L}\frac{dx'}{x'-x}+\frac{B_0}{\pi}\int_{L}^{2L}\frac{dx'}{x'-x}$$

$$=\frac{-B_0}{\pi}\ln\left(\left|\frac{x+L}{x+2L}\right|\right)+\frac{B_0}{\pi}\ln\left(\left|\frac{x-2L}{x-L}\right|\right)=\frac{B_0}{\pi}\ln\left(\left|\frac{x^2-4L^2}{x^2-L^2}\right|\right)$$

For given imaginary function, this is the form of the real function.

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x}dx'\right)$$

For our example: $$a(x) = \frac{B_0}{\pi}\ln\left(\left|\frac{4L^2 - x^2}{L^2 - x^2}\right|\right)$$



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Summary of results.

Summary

For a function $f(x)$, analytic along the real line:

$$f(x)=\Re\left(f(x)\right)+i\Im\left(f(x)\right)=a\left(x\right)+ib\left(x\right)$$

$$\Rightarrow a\left(x\right)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx' \qquad b\left(x\right)=-\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a\left(x'\right)}{x'-x}dx'$$

Example:

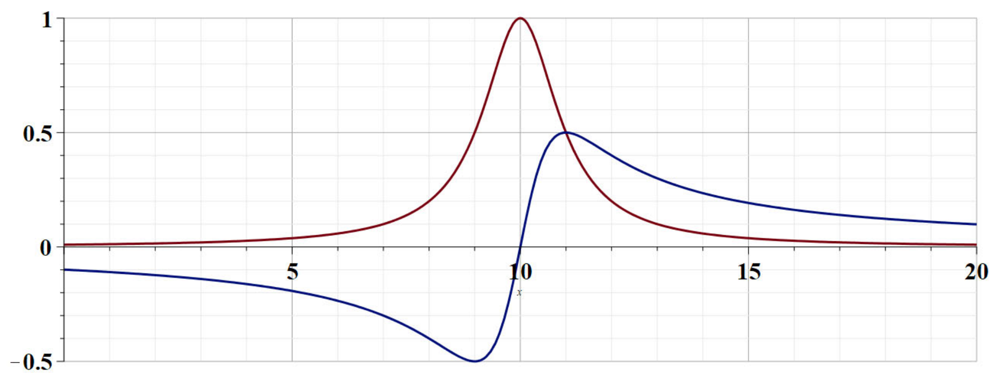
$$f(x)=\frac{1}{x+i} \qquad a(x)=\frac{x}{x^2+1} \qquad b(x)=-\frac{1}{x^2+1}$$

Check:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b\left(x'\right)}{x'-x}dx'=-\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{\left(x'-x\right)\left(x'^2+1\right)}dx'\overset{?}{=}\frac{x}{x^2+1}=a(x)$$

Summary.

$$a(\omega) = \frac{\omega - 10}{(\omega - 10)^2 + 1} \qquad b(\omega) = \frac{1}{(\omega - 10)^2 + 1}$$



Continued:

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x'-x)(x'^2+1)}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(x'-x)(x'^2+1)} - \frac{1}{(x'-x)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= -\frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x^2-x'^2}{(x'-x)(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

$$= \frac{P}{\pi}\int_{-\infty}^{\infty}\left(\frac{x+x'}{(x'^2+1)(x^2+1)}\right)dx' - \frac{1}{(x^2+1)}\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx'$$

Note that: $\int_{x+\epsilon}^{X}\frac{1}{x'-x}dx' = \ln(X-x) - \ln(\epsilon) = \ln\left(\frac{X-x}{\epsilon}\right)$

$$\int_{-X}^{x-\epsilon}\frac{1}{x'-x}dx' = -\ln(-X-x) + \ln(-\epsilon) = -\ln\left(\frac{X+x}{\epsilon}\right)$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'-x}dx' = \lim_{X\to\infty}\ln\left(\frac{X-x}{X+x}\right) = 0 \qquad \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{1}{x'^2+1}dx' = 1$$

$$\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' = \frac{x}{x^2+1} = a(x)$$