

**PHY 711 Classical Mechanics and
Mathematical Methods**

10-10:50 AM MWF in Olin 103

**Notes for Lecture 24: Rigid bodies –
Chap. 5 (F &W)**

- 1. Rigid body motion**
- 2. Moment of inertia tensor**
- 3. Torque free motion**

12	Fri, 9/17/2021	Chap. 3 & 6	Hamiltonian equations of motion	#9	9/20/2021
13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorem	#10	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap. 1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
 24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021

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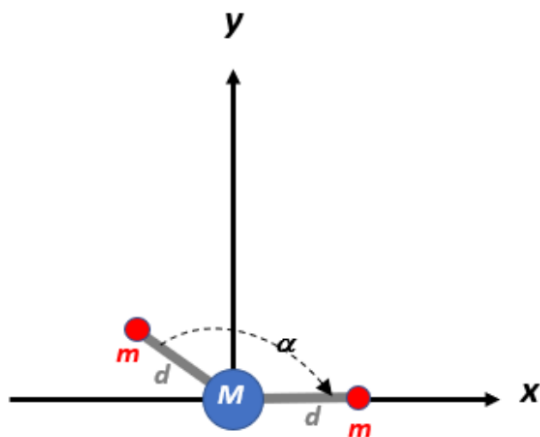
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In this lecture we will consider the rotational motion of rigid bodies as presented in Chapter 5 of your textbook.

PHY 711 -- Assignment #17

Oct. 22, 2021

Start reading Chapter 5 in Fetter & Walecka.



1. The figure above shows a rigid 3 atom molecule placed in the x - y plane as shown. Assume that the rigid bonds are massless.
 - a. Find the moment of inertia tensor in the given coordinate system placed of mass M in terms of the atom masses, bond lengths d , and angle α .
 - b. Find the principal moments of inertia I_1, I_2, I_3 and the corresponding principal axes.

(Extra credit.) Find the principal moments and axes for a coordinate system centered at the center of mass of the molecule.

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The physics of rigid body motion; body fixed frame vs inertial frame; results from Chapter 2:

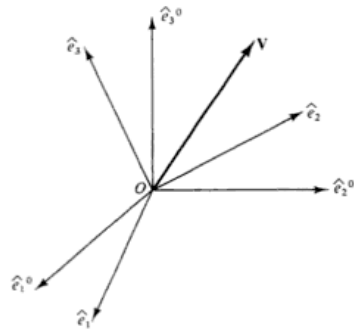


Figure 6.1 Transformation to a rotating coordinate system.

Let $\mathbf{V}$ be a general vector, e.g., the position of a particle. This vector can be characterized by its components with respect to either orthonormal triad. Thus we can write

$$\mathbf{V} = \sum_{i=1}^3 V_i^0 \hat{e}_i^0 \quad (6.1a)$$

$$\mathbf{V} = \sum_{i=1}^3 V_i \hat{e}_i \quad (6.1b)$$

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Determining the relationship between the inertial and body frames.

Recall from Chapter 2 -- Comparison of analysis in “inertial frame” versus “non-inertial frame”

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

For an arbitrary vector $\mathbf{V}$: $\mathbf{V} = \sum_{i=1}^3 V_i^0 \hat{e}_i^0 = \sum_{i=1}^3 V_i \hat{e}_i$

$$\left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \sum_{i=1}^3 \frac{dV_i^0}{dt} \hat{e}_i^0 = \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define: } \left(\frac{d\mathbf{V}}{dt} \right)_{body} \equiv \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i$$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

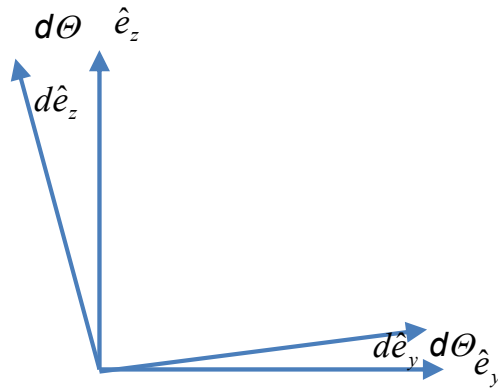
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Recall our previous discussion of rotating frames of reference.

Properties of the frame motion (rotation):



$$d\hat{e}_y = d\Theta \hat{e}_z$$

$$d\hat{e}_z = -d\Theta \hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\Theta \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Theta}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

$$\begin{pmatrix} d\hat{e}_y \\ d\hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos(d\Theta) & \sin(d\Theta) \\ -\sin(d\Theta) & \cos(d\Theta) \end{pmatrix} \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} - \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Theta \\ -d\Theta & 0 \end{pmatrix} \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

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Consider an infinitesimal rotation.

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration:

$$\left(\frac{d}{dt} \frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right) \left\{ \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V} \right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

Rotational acceleration.

Kinetic energy of rigid body :

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r}$$

=0 for rigid body

$$\rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\begin{aligned} T &= \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p \right)^2 \\ &= \sum_p \frac{1}{2} m_p (\boldsymbol{\omega} \times \mathbf{r}_p) \cdot (\boldsymbol{\omega} \times \mathbf{r}_p) \\ &= \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right] \end{aligned}$$



For a rigid body, internal motions are negligible.

$$T = \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) (\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \tilde{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Moment of inertia tensor:

$$\tilde{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p) \quad (\text{dyad notation})$$

Matrix notation :

$$\tilde{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

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It is convenient to group terms that depend on the body geometry – leading to the definition of the moment of inertia tensor.

Moment of inertia tensor:

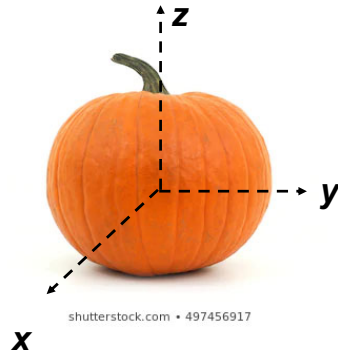
$$\vec{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p) \quad (\text{dyad notation})$$

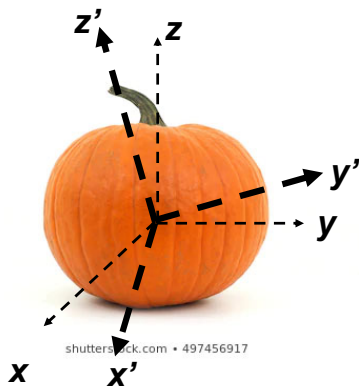
Note: For a given object and a given coordinate system, one can find the moment of inertia matrix

Matrix notation :

$$\vec{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$





Moment of inertia in
original coordinates

$$\vec{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

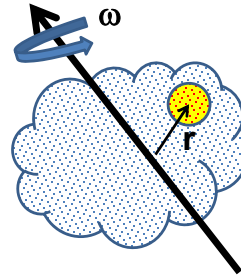
Moment of inertia in principal axes **(x',y',z')**

$$\vec{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Angular momentum of rigid body:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r}$$

$=0$ for rigid body



$$\Rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{L} = \sum_p \mathbf{r}_p \times (m_p \mathbf{v}_p) = \sum_p m_p \mathbf{r}_p \times (\boldsymbol{\omega} \times \mathbf{r}_p)$$

$$= \sum_p m_p (\boldsymbol{\omega} r_p^2 - \mathbf{r}_p (\boldsymbol{\omega} \cdot \mathbf{r}_p)) = \vec{\mathbf{I}} \cdot \boldsymbol{\omega}$$

$$\text{where } \vec{\mathbf{I}} \equiv \sum_p m_p (1r_p^2 - \mathbf{r}_p \mathbf{r}_p)$$

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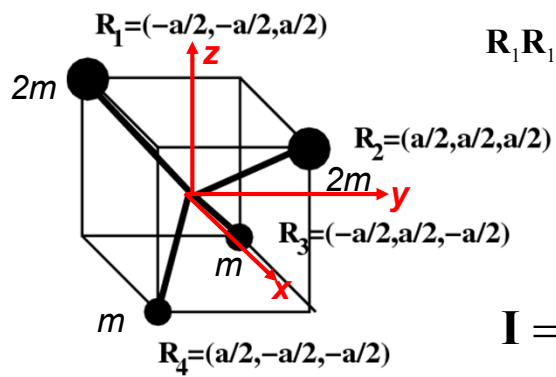
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The angular momentum can also be calculated.

An example with 4 point masses and massless rigid bonds

$$\vec{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p)$$

$$R_1^2 = R_2^2 = R_3^2 = R_4^2 = \frac{3a^2}{4}$$

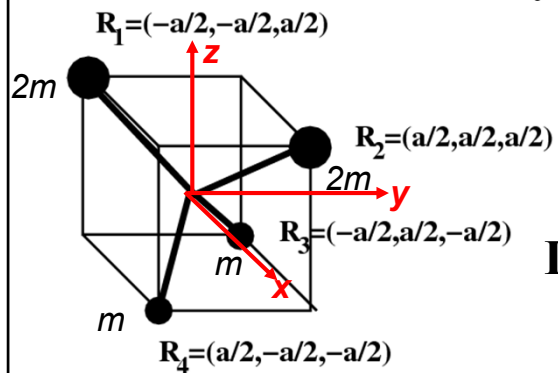


$$\mathbf{R}_1 \mathbf{R}_1 = \frac{a^2}{4} (-\hat{x} - \hat{y} + \hat{z})(-\hat{x} - \hat{y} + \hat{z})$$

$$\mathbf{I} = ma^2 \begin{pmatrix} 3 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Example continued --

$$\vec{I} \equiv \sum_p m_p (\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p)$$



$$\mathbf{I} = ma^2 \begin{pmatrix} 3 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$I_1 = \frac{7}{2} ma^2 \quad \mathbf{v}_1 = \sqrt{\frac{1}{2}} (\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

Principal moments: $I_2 = \frac{5}{2} ma^2 \quad \mathbf{v}_2 = \sqrt{\frac{1}{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}})$

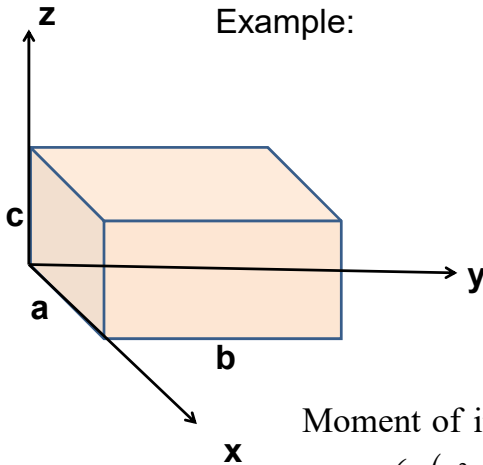
$$I_3 = 3ma^2 \quad \mathbf{v}_3 = \hat{\mathbf{z}}$$

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Example:



Moment of inertia tensor :

$$\tilde{\mathbf{I}} = M \begin{pmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2 + b^2) \end{pmatrix}$$

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Consider a simple rectangular solid with a coordinate system at the edge of the system.

Properties of moment of inertia tensor:

- Symmetric matrix → real eigenvalues I_1, I_2, I_3
- → orthogonal eigenvectors

$$\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$$

Moment of inertia tensor :

$$\vec{\mathbf{I}} = M \begin{pmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2 + b^2) \end{pmatrix}$$

For $a = b = c$:

$$I_1 = \frac{1}{6}Ma^2 \quad I_2 = \frac{11}{12}Ma^2 \quad I_3 = \frac{11}{12}Ma^2$$

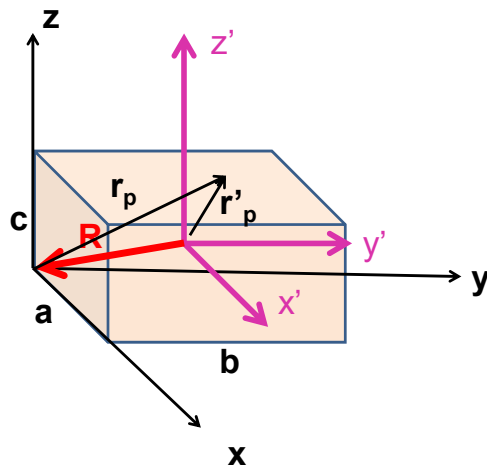
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The moment of inertia tensor in matrix form is a symmetric matrix and therefore can be diagonalized. The eigenvalues are known as principal moments of inertia and the eigenvectors are known as principal axes.

Changing origin of rotation



$$I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

$$I'_{ij} \equiv \sum_p m_p (\delta_{ij} r_p'^2 - r'_{pi} r'_{pj})$$

$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass :

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M(R^2 \delta_{ij} - R_i R_j) + M(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj})$$

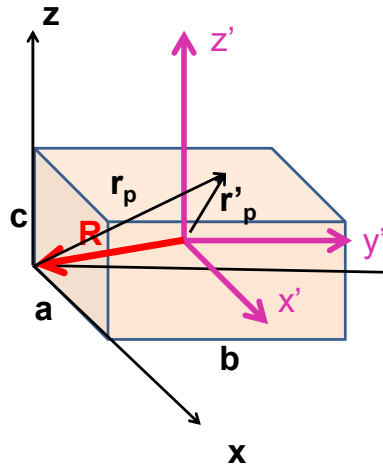
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Here we consider what happens when we evaluate the moment of inertia tensor about a different origin. In this case, the new origin happens to be at the center of mass.

$$I'_{ij} = I_{ij} + M(R^2 \delta_{ij} - R_i R_j) + M(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj})$$



Suppose that $\mathbf{R} = -\frac{a}{2}\hat{\mathbf{x}} - \frac{b}{2}\hat{\mathbf{y}} - \frac{c}{2}\hat{\mathbf{z}}$

and $\mathbf{r}_{CM} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M(R^2 \delta_{ij} - R_i R_j)$$

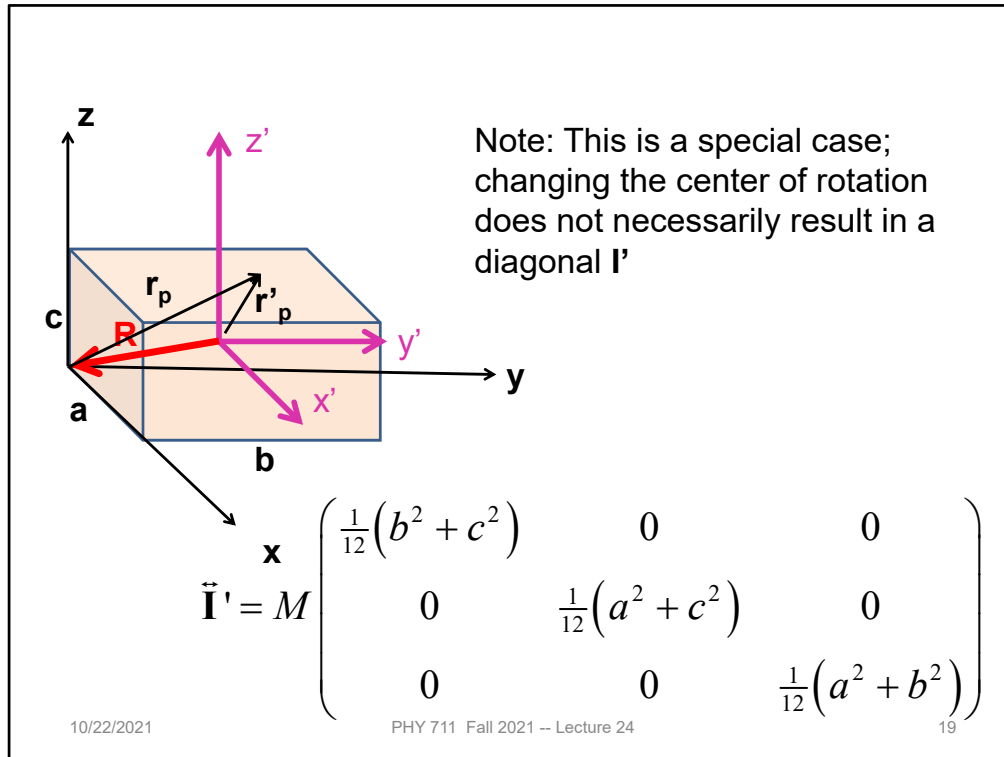
$$\begin{aligned} \vec{\mathbf{I}}' &= M \begin{pmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2 + b^2) \end{pmatrix} \\ &- M \begin{pmatrix} \frac{1}{4}(b^2 + c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}(a^2 + c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}(a^2 + b^2) \end{pmatrix} \end{aligned}$$

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Some details.



When the dust clears for this case. Note that I' happens to be diagonal already, however it is not generally true that shifting the origin for the moment of inertia would result in a diagonal matrix.

Descriptions of rotation about a given origin

For general coordinate system

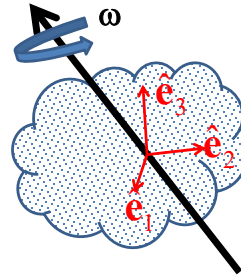
$$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$



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In the next slides we will focus on the fact that each rigid body has 3 principal axes and 3 moments of inertia for a given origin. It is often convenient to use that coordinate system to analyze rigid body motion.

Descriptions of rotation about a given origin -- continued
 Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

For (body fixed) coordinate system that diagonalizes
 moment of inertia tensor:

$$\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = & I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 \\ & + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 \end{aligned}$$

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Here we consider the angular moment expressed in the diagonalized body fixed frame of reference.

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations :

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = & I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 \\ & + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 = 0 \end{aligned}$$

While it is very difficult to express torque in this reference frame, we can readily solve problems with zero torque.

Torqueless Euler equations for rotation in body fixed frame:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

→ Solution for symmetric top -- $I_2 = I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_1) = 0$$

$$I_1 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 = 0 \quad \Rightarrow \quad \tilde{\omega}_3 = (\text{constant})$$

$$\text{Define : } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega$$

$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

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For the general system with three distinct moment of inertia, the solutions are difficult, but simplifications occur when two moments are the same, in this $I_1 = I_2$.

Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \quad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\begin{aligned} \text{Solution:} \quad \tilde{\omega}_1(t) &= A \cos(\Omega t + \phi) \\ \tilde{\omega}_2(t) &= A \sin(\Omega t + \phi) \\ \tilde{\omega}_3(t) &= \tilde{\omega}_3 \quad (\text{constant}) \end{aligned}$$

$$T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\begin{aligned} \mathbf{L} &= I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3 \\ &= I_1 A (\cos(\Omega t + \phi) \hat{\mathbf{e}}_1 + \sin(\Omega t + \phi) \hat{\mathbf{e}}_2) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3 \end{aligned}$$

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Time dependence of the symmetric top in free space

Torqueless Euler equations for rotation in body fixed frame:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

➔ Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Now consider the more general case.

Euler equations for rotation in body fixed frame :

$$I_1 \ddot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \ddot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \ddot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Approximate solution --

Suppose: $\ddot{\tilde{\omega}}_3 \approx 0$ Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

Reasonable approximations.

Euler equations for asymmetric top -- continued

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1 \tilde{\omega}_2 \quad \dot{\tilde{\omega}}_2 = \Omega_2 \tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A \cos(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$$\tilde{\omega}_2(t) \approx A \sqrt{\frac{\Omega_2}{\Omega_1}} \sin(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

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We see that there are conditions that allow stability for this system,. --- to be continued.