PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 24 – Chap. 5 (F &W)

Rotational motion

1. **Torque free motion of a rigid body**
2. **Rigid body motion in body fixed frame**
3. **Conversion between body and inertial reference frames**
4. **Symmetric top motion**

In this lecture, we continue our discussion of rigid body motion.

13	Mon, 9/20/2021	Chap. 3 & 6	Liouville theorm	#10	9/22/2021
14	Wed, 9/22/2021	Chap. 3 & 6	Canonical transformations		
15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021
25	Mon, 10/25/2021	Chap. 5	Rigid body motion	#18	10/29/2021

10/19/2020 PHY 711 Fall 2020 -- Lecture 24 2

Since you will be turning in your exams today, we will resume the homework assignments.

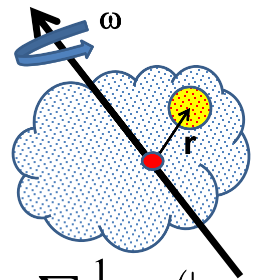
PHY 711 -- Assignment #18

Oct. 25, 2021

Continue reading Chapter 5 in **Fetter & Walecka**.

1. Work problem 5.9, parts (a) and (b) at the end of the chapter.

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega} \times \mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right)\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)^2\right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

Review of notions of rigid body motion.

Moment of inertia tensor

Matrix notation:

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

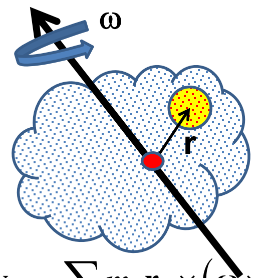
For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

In general there is a symmetric tensor which defines the moment of inertia. By rotating the coordinates about a fixed origin we can find the matrix in diagonal form.

Continued -- summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

Angular momentum: $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times (\boldsymbol{\omega}\times\mathbf{r}_p)$

$$\mathbf{L} = \sum_p m_p \left[\boldsymbol{\omega}(\mathbf{r}_p\cdot\mathbf{r}_p) - \mathbf{r}_p(\mathbf{r}_p\cdot\boldsymbol{\omega})\right]$$

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1}r_p^2 - \mathbf{r}_p\mathbf{r}_p\right)$$

In addition to the kinetic energy, the angular momentum also can be expressed in terms of the moment of inertia tensor.

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Time derivative: $\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2 \right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3 \right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1 \right) \hat{\mathbf{e}}_3$$

It is convenient to express the angular moment in terms of the principal moments .

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 +$$
$$\tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Want to determine angular velocities $\omega_i(t)$

When there is zero torque acting on the system, the angular velocity components are coupled through these Euler equations.

Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for symmetric top -- $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_1\right) = 0$$
$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define: $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$
$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

The solution to the coupled angular velocity components is in general complicated, but simplifies when two of the principal moments are equal for a "symmetric top".

Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\text{Solution :} \quad \tilde{\omega}_1(t) = A\cos(\Omega t + \varphi)$$

$$\tilde{\omega}_2(t) = A\sin(\Omega t + \varphi)$$

$$T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\begin{aligned}\mathbf{L} &= I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3 \\ &= I_1 A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3\end{aligned}$$

Details of the solution for a symmetric top.

Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

For example, the object starts spinning along the 3 axis. Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

Now consider the case where all of the principal moments are unequal.

Euler equations for asymmetric top -- continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\qquad \Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

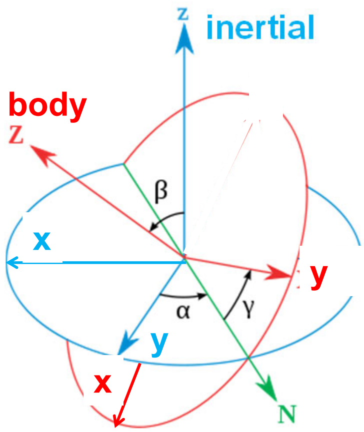
$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

We can find approximate stable solutions in certain cases.

Transformation between body-fixed and inertial coordinate systems – Euler angles



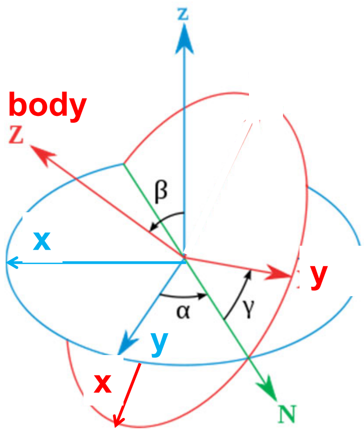
Comment – Since this is an old and intriguing subject, there are a lot of terminologies and conventions, not all of which are compatible. We are following the convention found in most quantum mechanics texts and NOT the convention found in most classical mechanics texts. Euler's main point is that any rotation can be described by 3 successive rotations about 3 different (not necessarily orthogonal) axes. In this case, one is along the inertial z axis and another is along the body fixed Z axis. The middle rotation is along an intermediate N axis.

http://en.wikipedia.org/wiki/Euler_angles

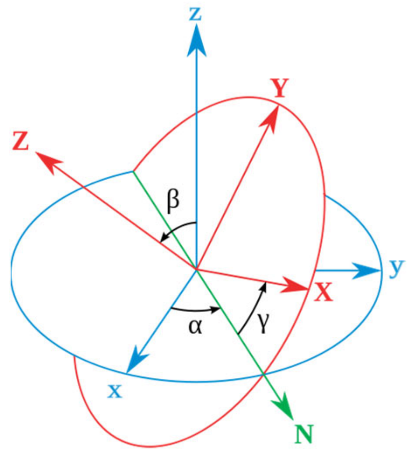
In order to consider motion of a rigid body more generally, in the presence of torque, it will be necessary to consider how to relate the body – fixed coordinates that diagonalize the moment of inertia tensor to another coordinate system which in general be an inertial coordinate system. Here again, we use ideas of Euler. This notation or it is equivalent is typically consistent with quantum mechanics text books.

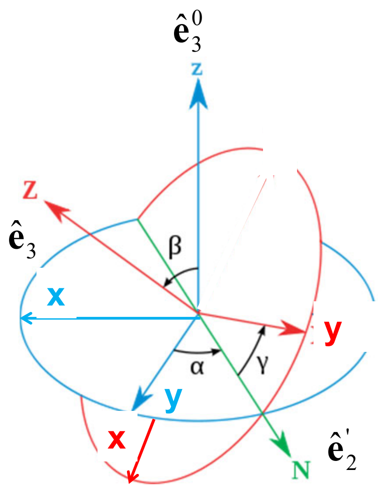
Comment on conventions

Our diagram



On web (for CM)





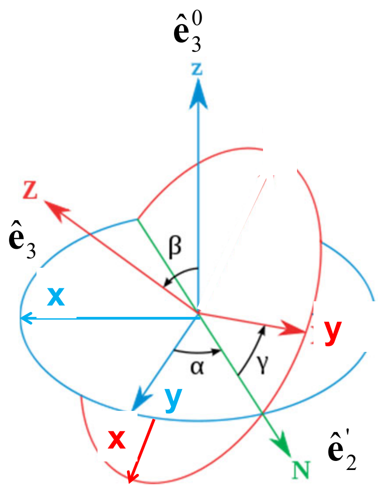
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

Euler said that the transformation of body-fixed → inertial frames can be accomplished in 3 steps and the corresponding angles are alpha, beta, and gamma. In this case, we want to express all results in the body fixed frame.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



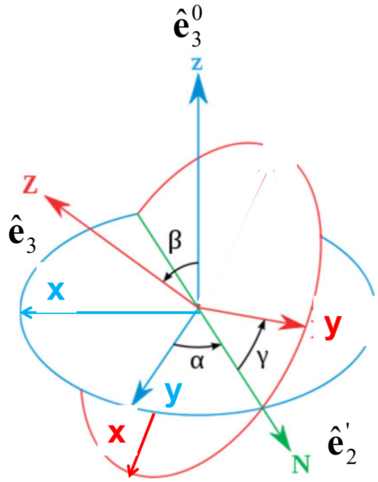
$$\hat{\mathbf{e}}_2^{'} = \sin\gamma\;\hat{\mathbf{e}}_1 + \cos\gamma\;\hat{\mathbf{e}}_2$$

Matrix representation:

$$\hat{\mathbf{e}}_2^{'} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

We can express the angular velocities in terms of the time rate of change of the alpha, beta, and gamma Euler angles. We can also express the rotation axes in terms of the instantaneous Euler angles as well. Here is the transformation of the middle axis.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_3^0 = -\sin\beta\ \hat{\mathbf{e}}'_1 + \cos\beta\ \hat{\mathbf{e}}'_3$$
$$= -\cos\gamma\sin\beta\ \hat{\mathbf{e}}_1 + \sin\gamma\sin\beta\ \hat{\mathbf{e}}_2 + \cos\beta\ \hat{\mathbf{e}}_3$$

Matrix representation:

$$\hat{\mathbf{e}}_3^0 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

Here is the transformation of the inertial 3 axis.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\ \hat{\mathbf{e}}_3^0 + \dot{\beta}\ \hat{\mathbf{e}}_2^{'} + \dot{\gamma}\ \hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

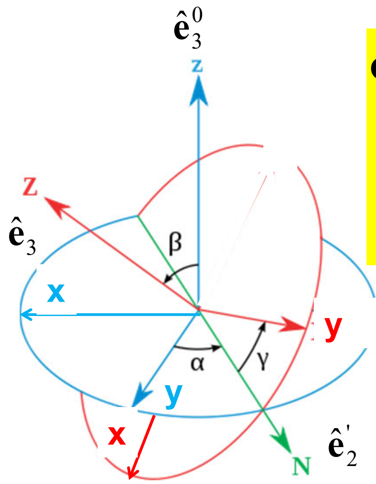
$$\tilde{\omega}_1 = \dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma$$

$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

Putting all of the transformations together, we now have expressions for the angular velocity components referenced to the body fixed frame.

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\tilde{\boldsymbol{\omega}} = \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 + \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 + \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3$$

Result to remember.

Rotational kinetic energy

$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\tilde{\omega}_1^2+\frac{1}{2}I_2\tilde{\omega}_2^2+\frac{1}{2}I_3\tilde{\omega}_3^2$$

$$=\frac{1}{2}I_1\left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right)+\dot{\beta}\sin\gamma\right]^2$$

$$+\frac{1}{2}I_2\left[\dot{\alpha}\left(\sin\beta\sin\gamma\right)+\dot{\beta}\cos\gamma\right]^2$$

$$+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2$$

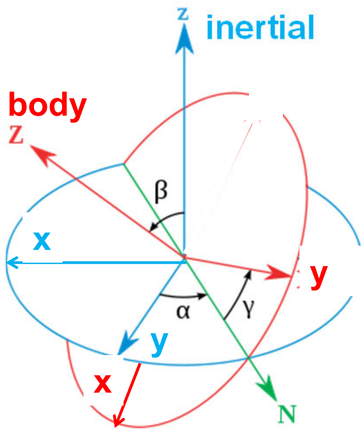
If $\quad I_1 = I_2$:

$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2$$

General expression of the rotational kinetic energy and the special case of the symmetric top.

Recap --

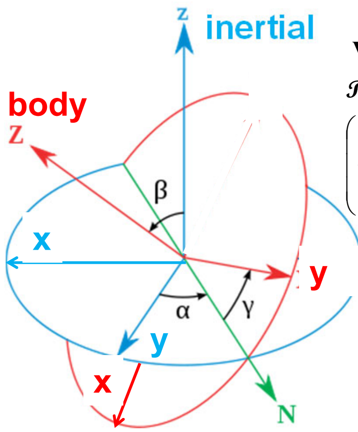
Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

In addition to the dynamic transformation needed for rigid body mechanics, this formalism is more generally useful when relating coordinate systems of different orientations.

General transformation between rotated coordinates – Euler angles



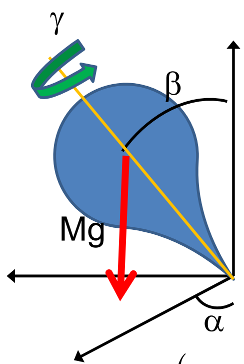
$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

In general any transformation can be expressed in terms of the three Euler angles.

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2} I_3 \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2 - Mgl\cos\beta$$

Now consider the motion of a symmetric top in which the pivot point is fixed and torque is applied by gravity acting at the center of mass of the top. Here l denotes the distance of the pivot point to the center of mass.

$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]^2 - Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]\cos\beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]$$

$$E = \frac{1}{2}I_1\dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

$$L(\beta,\dot{\beta}) = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} - Mgl\cos\beta$$

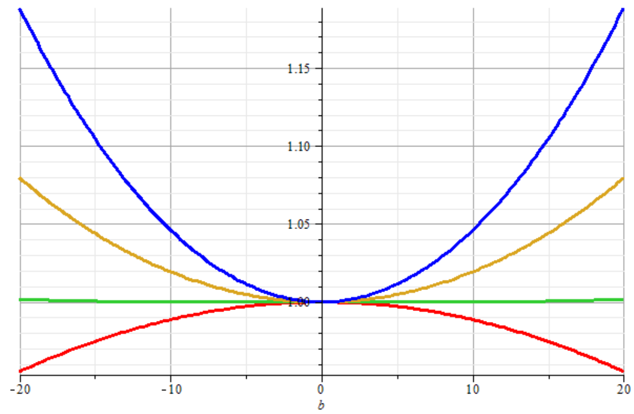
$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$

Lagrangian and its solution.

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

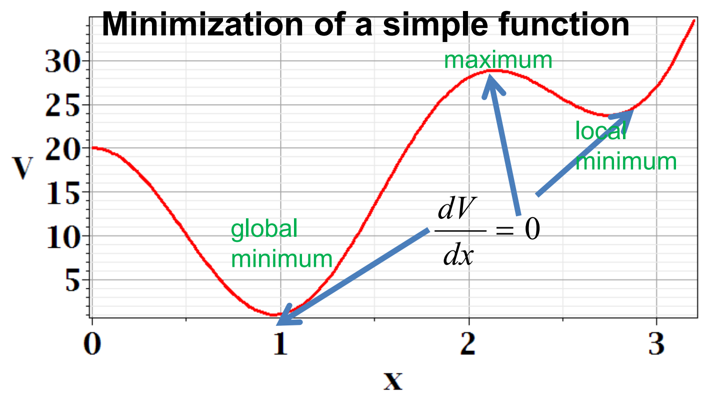
Stable/unstable solutions near β=0



Special case where top is spinning nearly vertically.

Your questions --How to decide stable/unstable solutions in slide 25? So for the problem on slide 23, if there is no initial movement/rotation of the top then the effective potential would stay as Mglcos(beta).

Comment – When we discussed one dimensional motion, we discussed stable and unstable equilibrium points. At equilibrium dV/dx=0, but only when V(x) has a minimum at that point, is the system stable in the sense that for small displacements from equilibrium, there are restoring forces to move the system back to the equilibrium point.



Suppose $\quad p_\alpha = p_\gamma \quad$ and $\quad \beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma{}^2}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \frac{1}{2}\beta^2\right)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma{}^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

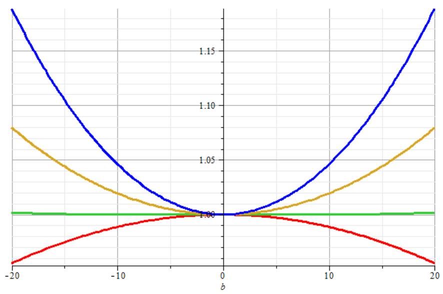
$\Rightarrow$ Stable solution if

$$p_\gamma \geq \sqrt{4MglI_1}$$

Note that

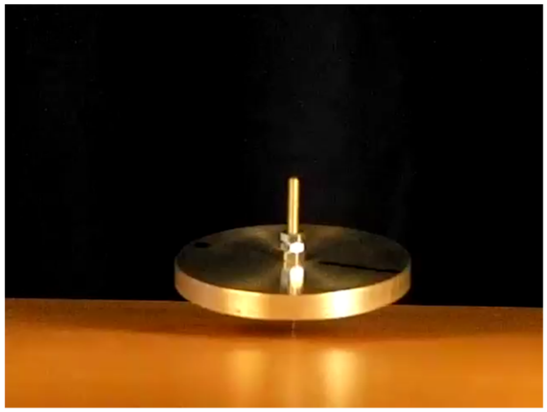
$p_\gamma = I_3\omega_3$

$\Rightarrow \omega_3$ must be sufficiently large for the top to maintain vertical orientation $(\beta \approx 0)$.



Approximate solution for that case.

http://www.physics.usyd.edu.au/~cross/SPINNING%20TOPS.htm



See also --

Home > American Journal of Physics > Volume 81, Issue 4 > 10.1119/1.4776195

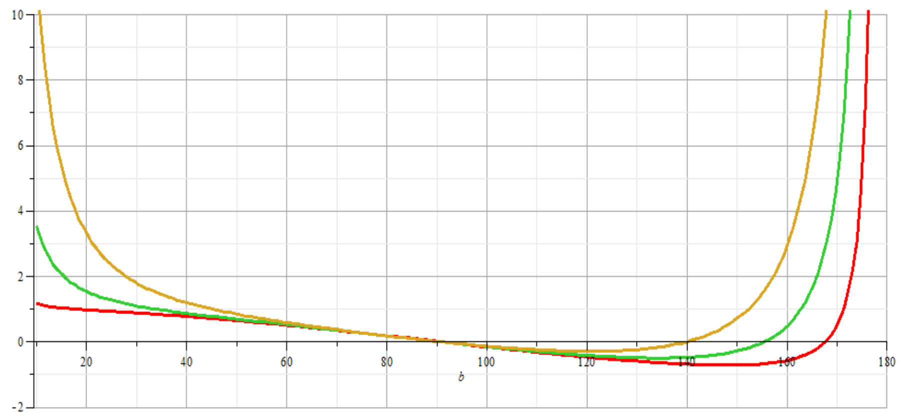
Full . Published Online: 18 March 2013 Accepted: December 2012

The rise and fall of spinning tops

American Journal of Physics **81**, 280 (2013); https://doi.org/10.1119/1.4776195

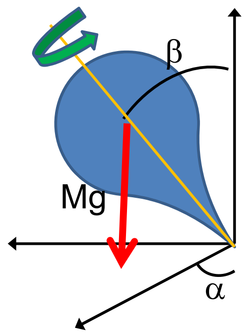
More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$



10/19/2020 PHY 711 Fall 2020 -- Lecture 24 29

For a spinning bicycle suspended by a rope, the beta angle can be greater than 90 degress https://drive.google.com/file/d/0B14RyYwpwSDNcXdxTWI3OExHX1k/view



Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\cos\beta$$

$$= I_1\dot{\alpha}\sin^2\beta + p_\gamma\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$

Summary of results.