PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103
Notes for Lecture 26 – Chap. 8 (F & W)

Motions of elastic membranes

1. **Review of standing waves on a string**
2. **Standing waves on a two dimensional membrane.**
3. **Boundary value problems**

In this lecture, we will resume our consideration of elastic media, extending the one dimensional analysis of a string to a two dimensional membrane.

15	Fri, 9/24/2021	Chap. 4	Small oscillations about equilibrium	#11	9/27/2021
16	Mon, 9/27/2021	Chap. 4	Normal modes of vibration	#12	9/29/2021
17	Wed, 9/29/2021	Chap. 4	Normal modes of more complicated systems	#13	10/04/2021
18	Fri, 10/01/2021	Chap. 7	Motion of strings	#14	10/06/2021
19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021
25	Mon, 10/25/2021	Chap. 5	Rigid body motion	#18	10/29/2021
26	Wed, 10/27/2021	Chap. 8	Elastic two-dimensional membranes		

The homework assignment relate to rigid body motion.

Thursday’s Physics Colloquium

https://www.physics.wfu.edu/events/colloquium-inhomogeneous-superconductivity-pulsed-magnetic-fields-and-an-abundance-of-data-october-28-2021-at-4-pm/

PHYSICS COLLOQUIUM

THURSDAY
•
OCTOBER 28, 2021

4 PM in Olin 101

“Inhomogeneous Superconductivity, Pulsed Magnetic Fields, and an Abundance of Data”

Superconductivity, more than one hundred years after its discovery, has no universal underlying microscopic, quantum mechanical, explanation, that we know of. It is the biggest unsolved problem in condensed matter physics. At Clark University, we study various correlated electron states using high magnetic fields, low temperatures, and high pressures, with the hope that we can contribute to the understanding of how electrons behave in metals. After an introduction, this talk will concentrate on the subject of inhomogeneous superconductivity. The story begins in 1960 when Clogston and Chandrasekhar claimed there was an ultimate magnetic field that would destroy superconductivity, when the energy to flip an electron spin in a magnetic field (Zemann energy) exceeded the binding energy of

Charles C. Agosta, Ph.D.
Director of the 3/2 Engineering Program and Former Chair of the Department of Physics
Clark University

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Thursday’s lecture is by a Professor from Clark U in Massachusetts. Please consult the webpage for details including relevant references on the topic.

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Review of the wave equation in one special dimension.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

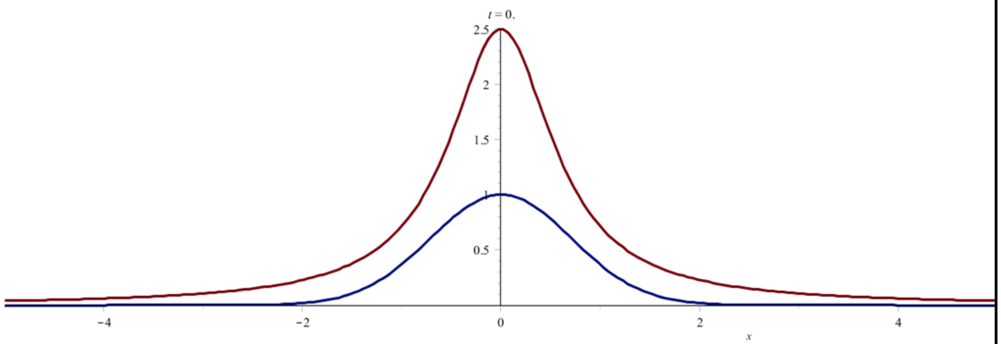
$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x')dx'$$

Review continued.

Example with $\psi(x) = 0$ and $\phi(x) = \dfrac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

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Two examples of traveling waves.

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

with $\mu(0,t) = \mu(L,t) = 0.$

Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\frac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0$ $\quad k = \frac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$

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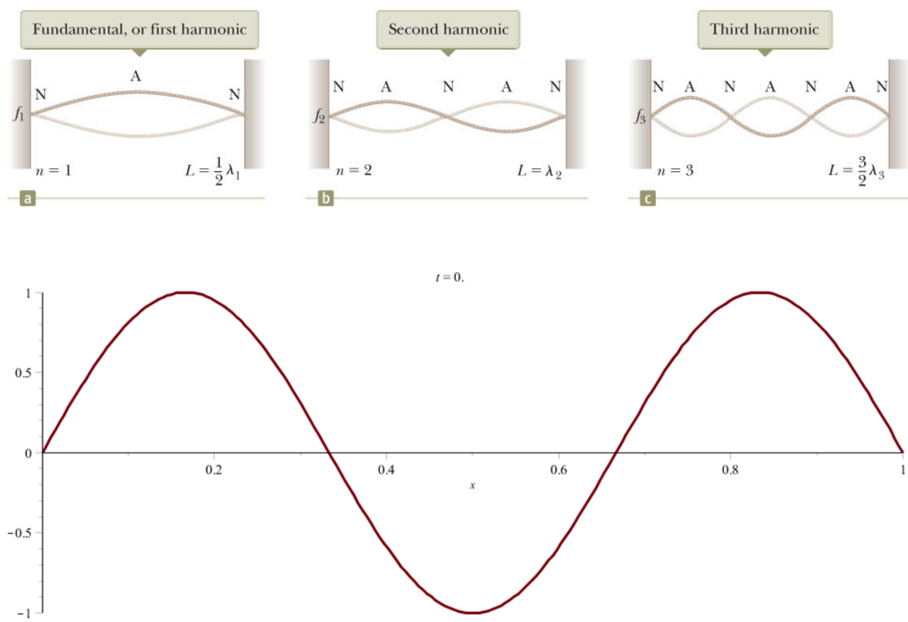
Standing wave solutions for constrained string.



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Some more details of standing waves.

Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

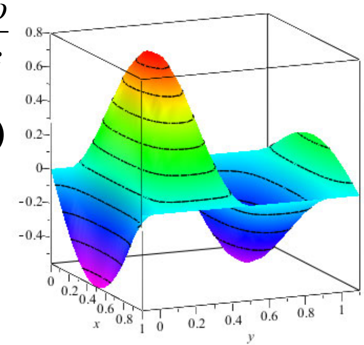
$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

$\rho(x,y)$



Note that here we are visualizing transverse waves. Longitudinal waves can also exist.

Now consider, the same idea, generalized to two spatial dimensions. Here we will focus on standing wave solutions.

In this case, we have mapped the one dimensional elastic string to a two dimensional elastic membrane

$$\frac{\partial^2}{\partial x^2} \rightarrow \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(in Cartesian coordinates)}$$

Lagrangian density: $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dxdy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial y)} = 0$$

It is possible to formulate the treatment using a continuous Lagrangian.

Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\nabla u\right)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\partial u/\partial t\right)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\partial u/\partial x\right)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial\left(\partial u/\partial y\right)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

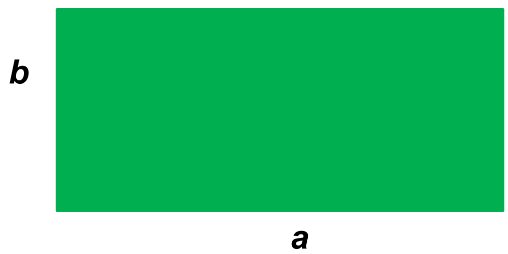
Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Some details.

Consider a rectangular boundary:



Clamped boundary conditions:

$$\rho(0,y) = \rho(a,y) = \rho(x,0) = \rho(x,b) = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0$$

$$\text{where } k = \frac{\omega}{c}$$

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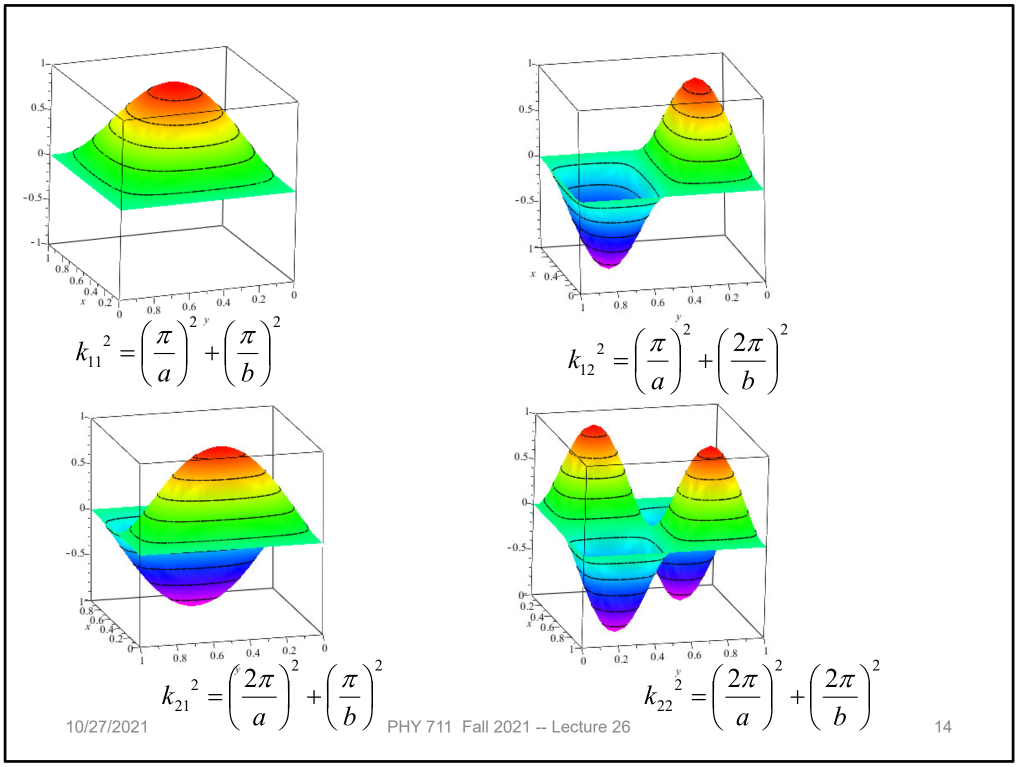
An example of the rectangular membrane clamped on all edges.



Some two dimensional standing waves.

More general boundary conditions:

$$\tau \nabla u\big|_b = \kappa u\big|_b \qquad \text{represents bounded side constrained with spring}$$

$$\tau \nabla u\big|_b = 0 \qquad \text{represents "free" side}$$

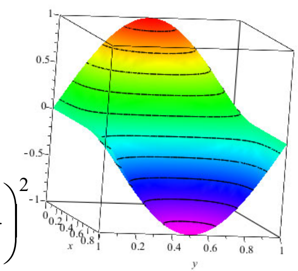
Mixed boundary conditions:

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

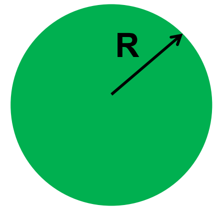
$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$k_{11}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



Other possible boundary conditions.

Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$\rho(R,\varphi)=0$

$$\left(\nabla^2+k^2\right)\rho(r,\phi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\Phi(\varphi)=e^{im\varphi}$

Note: $\Phi(\varphi)=\Phi(\varphi+2\pi)$

$\Rightarrow m=\text{integer}$

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Another example of a membrane, this time clamped at the boundary of a circle. (Such as in a drum for example.) It is convenient to polar coordinates.

Consider circular boundary -- continued

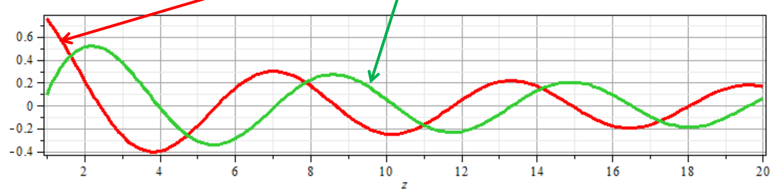
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



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The radial equation has this special form which is conveniently expressed in terms of Bessel functions.

Some properties of Bessel functions

Asending series: $J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!}\left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$

$$J_{m-1}(z) - J_{m+1}(z) = 2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z) \xrightarrow[z>>1]{} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$

Zeros of Bessel functions $J_m(z_{mn}) = 0$

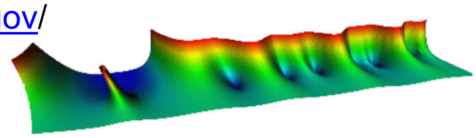
$m = 0$: $z_{0n} = 2.406,\ 5.520,\ 8.654, \ldots$

$m = 1$: $z_{1n} = 3.832,\ 7.016,\ 10.173, \ldots$

$m = 2$: $z_{2n} = 5.136,\ 8.417,\ 11.620, \ldots$

Some properties of Bessel functions of integer order.

http://dlmf.nist.gov/



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 *DLMF Update; Version 1.0.9*
2014-04-25 *DLMF Update; Version 1.0.8; errata & improved MathML*
2014-03-21 *DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics*
2013-08-16 *Bille C. Carlson, DLMF Author, dies at age 89*
More news

A resource for finding properties of special functions including Bessel functions.

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$Y_n(z) = -\frac{\left(\tfrac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\tfrac{1}{2}z\right) J_n(z)$$
$$-\frac{\left(\tfrac{1}{2}z\right)^{n}}{\pi} \sum_{k=0}^{\infty} \left(\psi(k+1)+\psi(n+k+1)\right) \frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!\,(n+k)!},$$

Some details.

Some properties of Bessel functions -- continued

Note: It is possible to prove the following identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)rdr = \frac{R^2}{2}\left(J_{m+1}\left(z_{mn}\right)\right)^2\delta_{nn'}$$

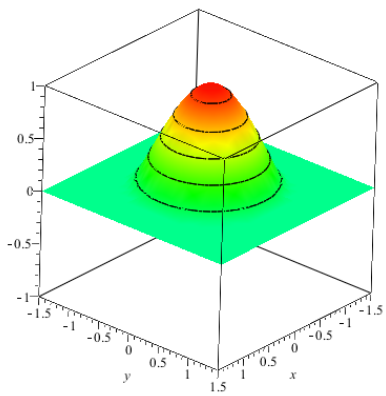
Returning to differential equation for radial function:

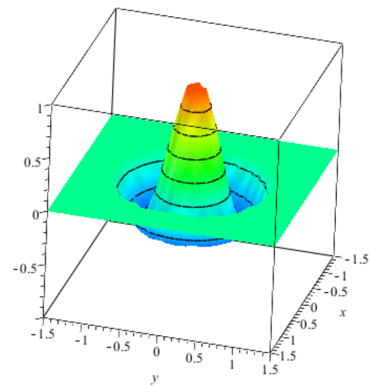
$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow\ f_{mn}(r) = AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn} = \frac{z_{mn}}{R}$$

Patient mathematicians worked out lots of useful relationships. We are particularly interested in aligning the zeros of the Bessel functions with the boundaries of our membrane.

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right) \qquad \rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$



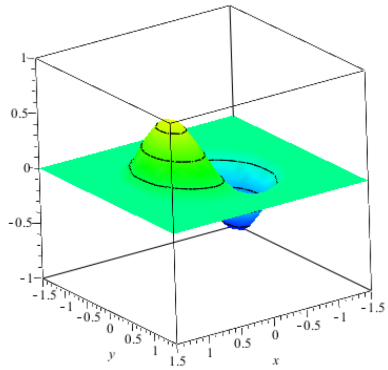


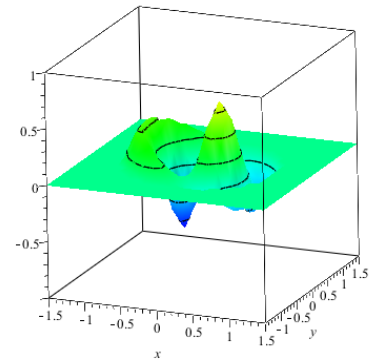
$$k_{01} = \frac{2.406}{R} \qquad k_{02} = \frac{5.520}{R}$$

Some examples.

$$\rho_{11}(r,\varphi) = f_{11}(r)\cos(\varphi) = AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi)$$

$$\rho_{12}(r,\varphi) = f_{12}(r)\cos(\varphi) = AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$





$$k_{11} = \frac{3.832}{R}$$

$$k_{12} = \frac{7.016}{R}$$

More examples.

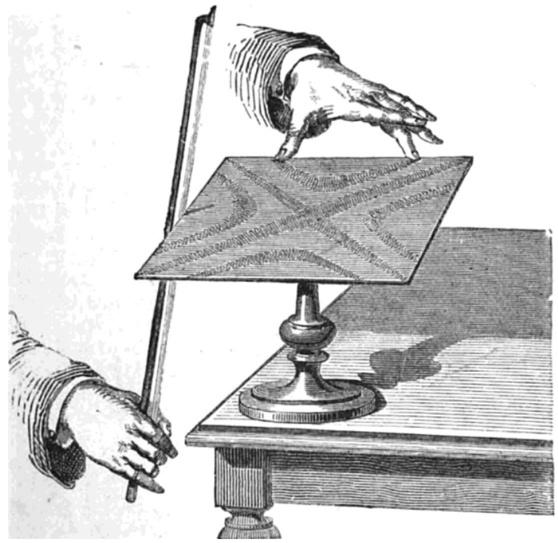
Ernst Chladni

Ernst Chladni

Born	30 November 1756 Wittenberg, Electorate of Saxony in the Holy Roman Empire
Died	3 April 1827 (aged 70) Breslau, Province of Silesia in the Kingdom of Prussia, a part of the German Confederation
Nationality	German
Known for	Study of acoustics Chladni plates and figures Estimating the speed of sound Chladni's law Theory of meteorites' origins

Scientific career

Fields	Physics



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A very nice demonstration of these standing waves was invented by Chladni

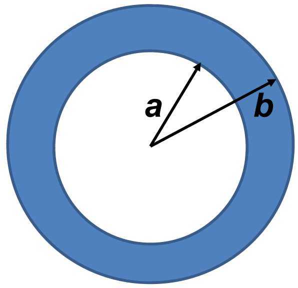
Demonstration with motor in the middle – (PASCO)

A picture of the demo we have in Olin.

http://www.physics.wfu.edu/resources/education-resources/demo-videos/waves/

Movie thanks to Eric Chapman.

More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

A non-trivial example with two boundaries.

Consider circular boundary -- continued

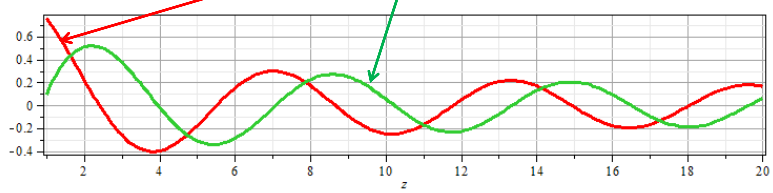
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

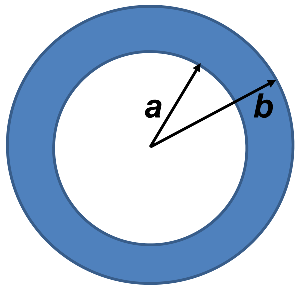
Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



In this case, both Bessel and Neumann functions are needed.

Normal modes of an annular membrane -- continued



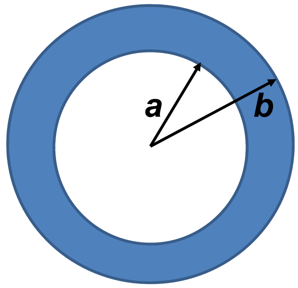
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

General form of radial function: $f(r)=AJ_m(kr)+BN_m(kr)$

We need to find the linear coefficients A and B and the wavevector k.

Normal modes of an annular membrane -- continued



Boundary conditions:

$f(a)=0 \qquad f(b)=0$

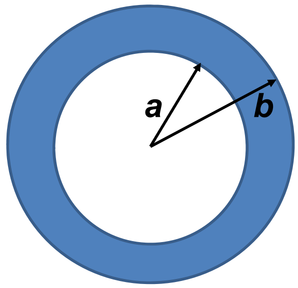
$$AJ_m(ka)+BN_m(ka)=0$$
$$AJ_m(kb)+BN_m(kb)=0$$

$\Rightarrow$ 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A}=\frac{-J_m(ka)}{N_m(ka)}=\frac{-J_m(kb)}{N_m(kb)} \quad \text{(transcendental equation for } k\text{)}$$

A method of solving this problem.

Normal modes of an annular membrane -- continued



Boundary conditions:

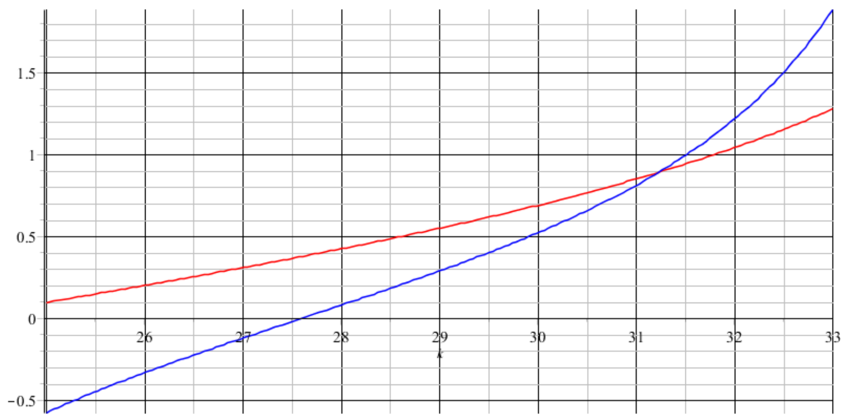
$f(a) = 0 \qquad f(b) = 0$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)}$$ -- in terms of solution k_{mn} :

$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right)$$

Analysis for *m=0* and a=0.1, b=0.2:

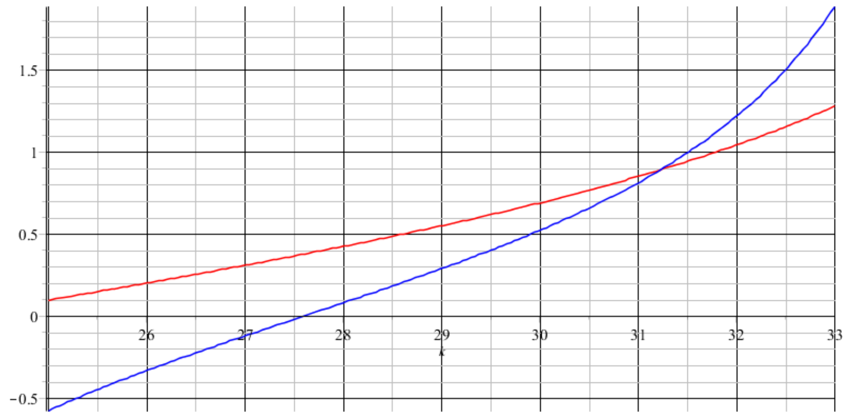
> $plot\left(\left\{\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)}, \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}\right\}, k=25\,..33, color=[red, blue]\right);$

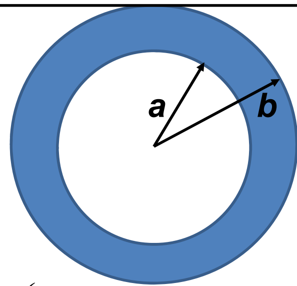


Finding a solution graphically.

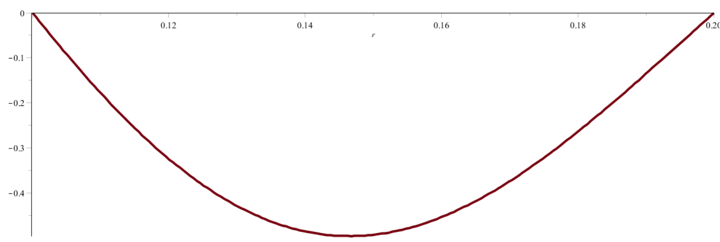
$$> fsolve\left(\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)} = \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}, k, 30 ..33\right);$$

31.23030920





$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right) \qquad k_{01} = 31.230309$$



Solution for this case.