

**PHY 711 Classical Mechanics and
Mathematical Methods**

10-10:50 AM MWF in Olin 103

Notes for Lecture 27 – Chap. 9 in F & W

Introduction to hydrodynamics

- 1. Motivation for topic**
- 2. Newton's laws for fluids**
- 3. Conservation relations**

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In this lecture we will begin an introductory treatment of the mechanics of fluids.

19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap. 1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021
25	Mon, 10/25/2021	Chap. 5	Rigid body motion	#18	10/29/2021
26	Wed, 10/27/2021	Chap. 8	Elastic two-dimensional membranes		
27	Fri, 10/29/2021	Chap. 9	Mechanics of 3 dimensional fluids		
28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids		



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Hydrodynamic analysis

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

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Here is a list of topics that will be covered in the next few lectures.

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$
 Pressure $p(x,y,z,t)$
 Velocity $\mathbf{v}(x,y,z,t)$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

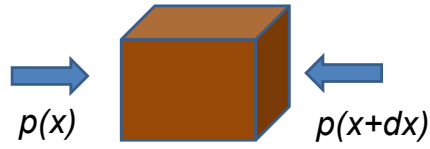
$$\mathbf{F} \rightarrow \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$$

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Newton's laws need to be adapted to describe the physics of fluids. Here pressure is important and more generally, the functions used to describe fluids depend on position and time.



$$\begin{aligned}
 F_{\text{pressure}}|_x &= (-p(x+dx, y, z) + p(x, y, z))dydz \\
 &= \frac{(-p(x+dx, y, z) + p(x, y, z))}{dx} dx dy dz \\
 &= -\frac{\partial p}{\partial x} dV
 \end{aligned}$$

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Pressure acts in all directions. Here we argue that the spatial derivative of the pressure applies a force to a volume of fluid.

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{applied}} \rho dV - (\nabla p) dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$m = \rho dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

It is convenient to write Newton's law in terms of the mass density, velocity, and pressure of the fluid.

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that : $\mathbf{v} \equiv v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}$

$$\frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z = \nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

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Because of the continuous nature of the velocity, the total time derivative of the fluid velocity depends both on the partial derivatives with respect to space and with respect to time as derived here.

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \left((\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\rho \left(\nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Some alternative expressions for the velocity terms.

Your question – What is irrotational flow?

Irrotational flow: $\nabla \times \mathbf{v} = 0$

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

Which of the following vector functions have zero curl?

- a. $\mathbf{v} = C\hat{\mathbf{x}}$ (C is a constant)
- b. $\mathbf{v} = Cx\hat{\mathbf{x}}$
- c. $\mathbf{v} = Cy\hat{\mathbf{x}}$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi \quad \Phi \text{ is "velocity potential"}$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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The restricted equations have some interesting properties.

Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C(t)$$

$$\text{where } \mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla (\Phi(\mathbf{r}, t) + C'(t))$$

$$\Rightarrow \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = 0 \quad \text{Bernoulli's theorem}$$

This result is known as Bernoulli's equation

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = 0$$

Modified form; assuming $\frac{\partial\Phi}{\partial t} = 0$

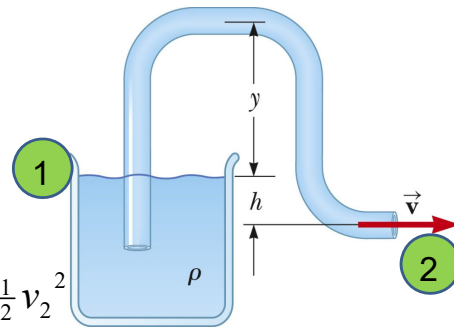
$$\frac{p}{\rho} + U + \frac{1}{2}v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$



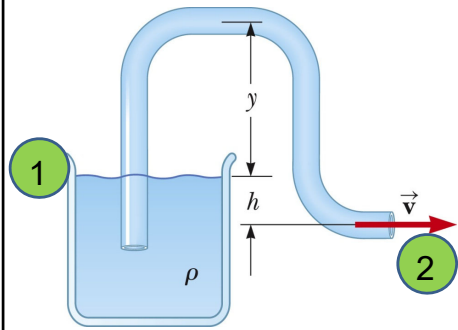
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This is a problem illustrating Bernoulli's equation as a syphon.

Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

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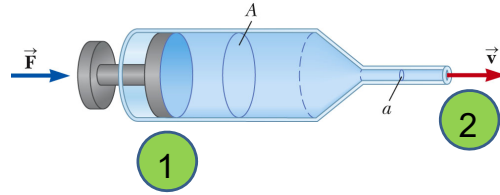
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This example is taken from the PHY 114 textbook

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \quad p_2 = p_{atm}$$

$$U_1 = U_2$$

$$v_1 A = v_2 a \quad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

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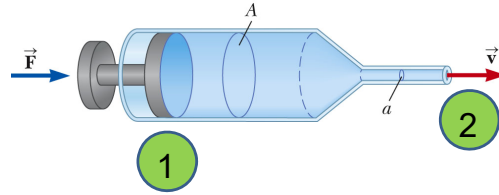
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Another example of Bernoulli's equation for a syringe.

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^2 \left(1 - \left(\frac{a}{A} \right)^2 \right)$$

$$v_2 = \sqrt{\frac{2F / A}{1 - \left(\frac{a}{A} \right)^2}}$$

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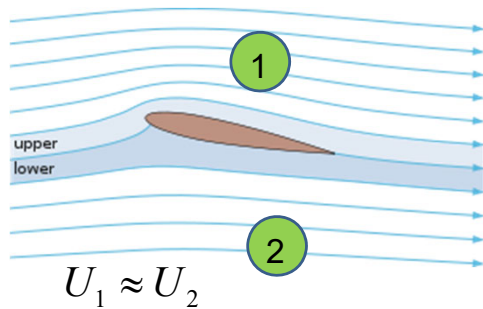
Syringe fluid continued.

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$

$$p_2 - p_1 = \frac{1}{2}\rho(v_1^2 - v_2^2)$$

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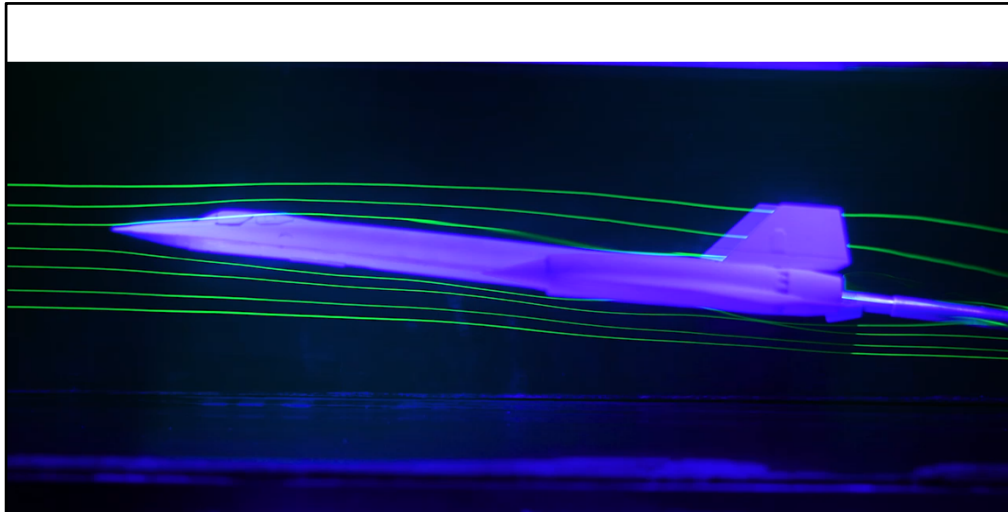
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This example of Bernoulli's equation is oversimplified. It appeared in most of the old textbook, but seems now to be deemphasized. It is given here since it shows some aspects of fluid flow, although apparently not good enough.

Your question -- What aspects do over simplified Bernoulli's equation not include in studying fluid dynamics?

According to a Scientific American article, the conclusion that $v_2 > v_1$ because of the shape of the airplane wing is not quite true. Numerical modeling reveal a more complicated picture.

<https://www.scientificamerican.com/article/no-one-can-explain-why-planes-stay-in-the-air/>



At NASA Ames Fluid Mechanics Laboratory, streamlines of dye in a water channel interact with a model airplane. Credit: *Ian Allen* (copied from Scientific American page mentioned above).

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \quad \begin{array}{l} \text{alternative form} \\ \text{of continuity equation} \end{array}$$

The continuity equation is an important aspect of fluid flow.

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid : $\rho = (\text{constant})$

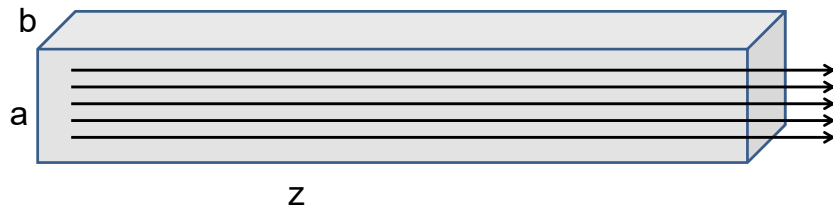
$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

$$\text{Irrotational flow : } \nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla \Phi$$

$$\Rightarrow \nabla^2 \Phi = 0$$

For an incompressible and irrotational fluid, it is mathematically convenient to express the velocity field in terms of a velocity potential field.

Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

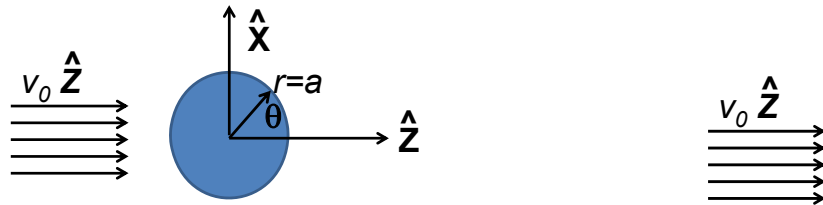
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For a uniformly fluid flowing along the z direction, the velocity potential and velocity field are easily written as shown.

Example – flow around a long cylinder (oriented in the Y direction)



$$\nabla^2 \Phi = 0$$

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

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Now consider the uniform fluid in the presence of an impediment. In this case we consider a cylindrical log.

Laplace equation in cylindrical coordinates

$(r, \theta, \text{defined in } x\text{-}z \text{ plane; } y \text{ representing cylinder axis})$

$$\nabla^2 \Phi = 0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition : $v_z(r \rightarrow \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \rightarrow \infty) = -v_0 \quad \Rightarrow \quad \Phi(r \rightarrow \infty, \theta) = -v_0 r \cos \theta$$

$$\text{Note that : } \frac{\partial^2 \cos \theta}{\partial \theta^2} = -\cos \theta$$

$$\text{Guess form : } \Phi(r, \theta) = f(r) \cos \theta$$

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We need to consider solutions of the Laplace equation.

Necessary equation for radial function

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \quad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

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Particular equations for this geometry and the application of the boundary values.

$$\Phi(r, \theta) = -v_0 \left(r + \frac{a^2}{r} \right) \cos \theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0 \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = v_0 \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2 \Phi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2}$$

to be continued ...

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More details.