PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 27 – Chap. 9 in F & W

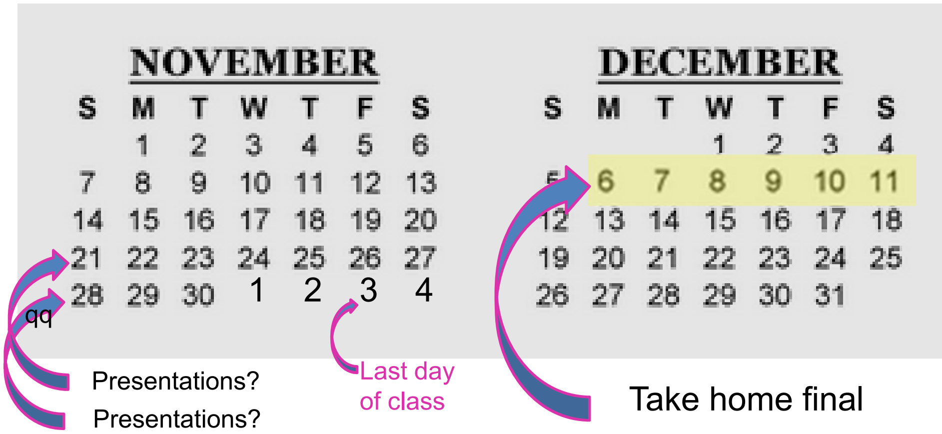
Introduction to hydrodynamics

1. **Motivation for topic**
2. **Newton's laws for fluids**
3. **Conservation relations**

In this lecture we will begin an introductory treatment of the mechanics of fluis.

19	Mon, 10/04/2021	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/06/2021	Chap.1-7	Review		
	Fri, 10/08/2021	No class	Fall break		
	Mon, 10/11/2021	No class	Take home exam		
	Wed, 10/13/2021	No class	Take home exam		
21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021
25	Mon, 10/25/2021	Chap. 5	Rigid body motion	#18	10/29/2021
26	Wed, 10/27/2021	Chap. 8	Elastic two-dimensional membranes		
27	Fri, 10/29/2021	Chap. 9	Mechanics of 3 dimensional fluids		
28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids		

Now is a good time to start thinking about your projects --



Project

The purpose of this assignment is to provide an opportunity for you to study a topic of your choice in greater depth. The general guideline for your choice of project is that it should have something to do with classical mechanics, and there should be some degree of of analytic or numerical computation associated with the project. The completed project will include a short write-up and a presentation to the class. You may design your own project or use one of the following list (which will be updated throughout the term).

- Consider a scattering experiment in which you specify the spherically symetric interaction potential V(r). Write a computer program (using your favorite language) to evaluate the scattering cross section for your system. (Depending on your choice, you may wish to present your results either in the the center-of-mass or lab frames of reference.)
- Consider the Foucoult Pendulum. Analyze the equations of motion including both the horizontal and vertical motions. You can either solve the equations exactly or use perturbation theory. Compare the effects of the vertical motion to the effects of air friction.
- Consider a model system of 2 or more interacting particles with appropriate initial conditions, using numerical methods to find out how the system evolves in time and space. For few particles and special initial conditions this approach can be used to explore orbital mechanics. For many particles and random initial conditions, this approach can be used to explore statistical mechanics via molecular dynamics simulations.
- Examine the normal modes of vibration for a model system with 3 or more masses in 2 or 3 dimensions.
- Analyze the soliton equations beyond what was covered in class.

Hydrodynamic analysis
Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Here is a list of topics that will be covered in the next few lectures.

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$
$$\text{Pressure} \quad p(x,y,z,t)$$
$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

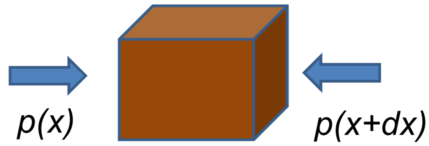
$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

Newton's laws need to be adapted to describe the physics of fluids. Here pressure is important and more generally, the functions used to describe fluids depend on position and time.



$$F_{pressure}\Big|_x = \left(-p(x+dx,y,z)+p(x,y,z)\right)dydz$$

$$= \frac{\left(-p(x+dx,y,z)+p(x,y,z)\right)}{dx}dxdydz$$

$$= -\frac{\partial p}{\partial x}dV$$

Pressure acts in all directions. Here we argue that the spatial derivative of the pressure applies a force to a volume of fluid.

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\, \rho dV - (\nabla p) dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$m = \rho dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

It is convenient to write Newton's law in terms of the mass density, velocity, and pressure of the fluid.

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that : $\mathbf{v} \equiv v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

$$\frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

Because of the continuous nature of the velocity, the total time derivative of the fluid velocity depends both or the partial derivates with respect to space and with respect to time as derived here.

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho\left(\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Some alternative expressions for the velocity terms.

Your question – What is irrotational flow?

Irrotational flow: $\nabla \times \mathbf{v} = 0$

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Which of the following vector functions have zero curl?

a. $\mathbf{v}=C\hat{\mathbf{x}}$ (C is a constant)

b. $\mathbf{v}=Cx\hat{\mathbf{x}}$

c. $\mathbf{v}=Cy\hat{\mathbf{x}}$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2} v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla \times \mathbf{v}\right) = 0$ "irrotational flow"

$\Rightarrow \mathbf{v} = -\nabla\Phi$ Φ is "velocity potential"

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2} v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2} v^2 - \frac{\partial \Phi}{\partial t}\right) = 0$$

The restricted equations have some interesting properties.

Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C'(t)\big)$

$$\Rightarrow \frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

This result is known as Bernoulli's equation

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0$$

Modified form; assuming $\frac{\partial \Phi}{\partial t} = 0$

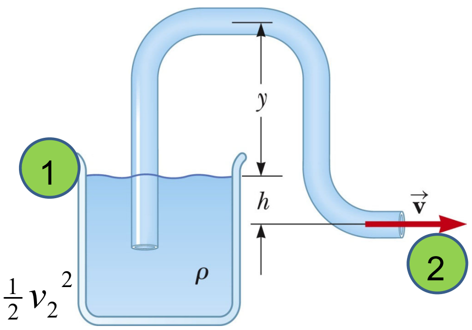
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

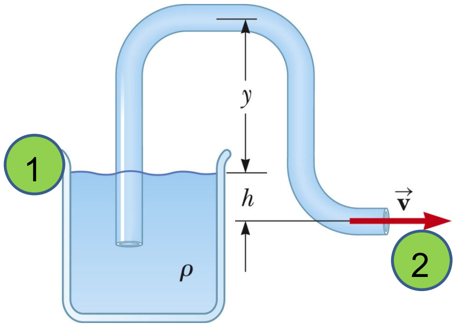
$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$



This is a problem illustrating Bernoulli's equation as a syphon.

Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

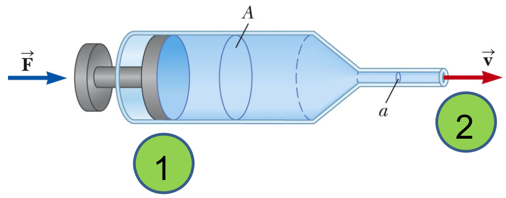
$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^{\,2}$$

$$v_2 \approx \sqrt{2gh}$$

This example is taken from the PHY 114 textbook

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2=\text{constant}$$



$$p_1=\frac{F}{A}+p_{atm} \qquad p_2=p_{atm}$$

$$U_1=U_2$$

$$v_1A=v_2a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}{v_1}^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}{v_2}^2$$

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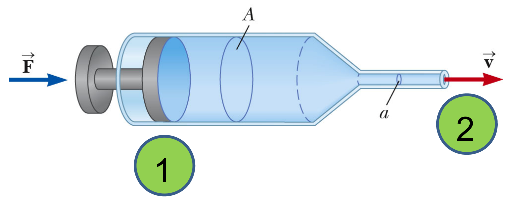
Another ezample of Bernoulli's equation for a syringe.

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2{}^2\left(1 - \left(\frac{a}{A}\right)^2\right)$$

$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

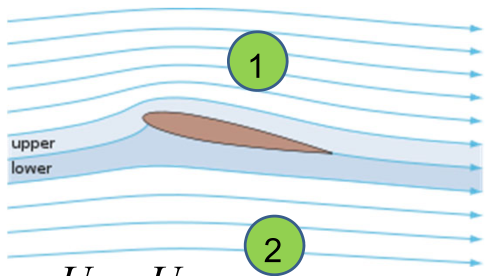
Syringe fluid continued.

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left({v_1}^2 - {v_2}^2\right)$$

This example of Bernoulli's equation is oversimplified. It appeared in most of the old textbook, but seems now to be deemphasized. It is given here since it shows some aspects of fluid flow, although apparently not good enough.

Your question -- What aspects do over simplified Bernoulli's equation not include in studying fluid dynamics?

According to a Scientific American article, the conclusion that $v_2 > v_1$ because of the shape of the airplane wing is not quite true. Numerical modeling reveal a more complicated picture.

https://www.scientificamerican.com/article/no-one-can-explain-why-planes-stay-in-the-air/

At NASA Ames Fluid Mechanics Laboratory, streamlines of dye in a water channel interact with a model airplane. Credit: *Ian Allen* (copied from Scientific American page mentioned above).

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0$ alternative form of continuity equation

The continuity equation is an important aspect of fluid flow.

Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

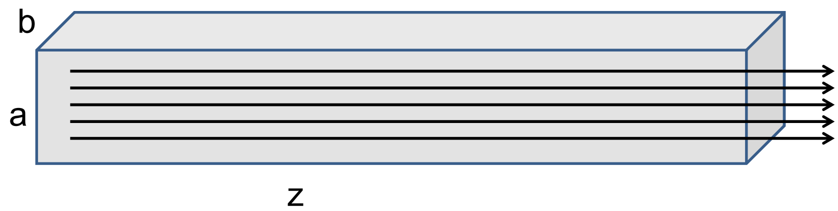
$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow: $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

For an incompressible and irrotational fluid, it is mathematically convenient to express the velocity field in terms of a velocity potential field.

Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

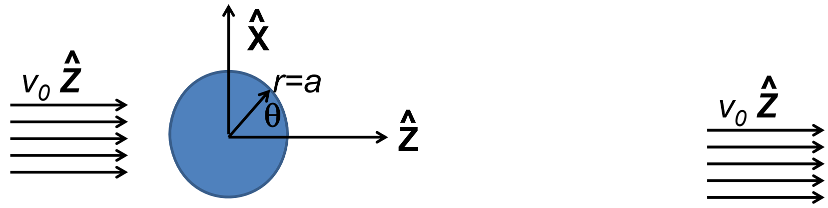
Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

For a uniformly fluid flowing along the z direction, the velocity potential and velocity field are easily written as shown.

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Now consider the uniform fluid in the presence of an impediment. In the is case we consider a cylindrical log.

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\frac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

We need to consider solutions of the Laplace equation.

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\quad \Rightarrow A = -v_0$

Particular equations for this geometry and the application of the boundary values.

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

to be continued …

More details.