PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 28 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Irrotational and incompressible fluids
3. Irrotational and isentropic fluids
4. Approximate solutions in the linear limit – next time

In this lecture, we will continue our discussion of hydrodynamics which is presented in Chapter 9 of your textbook.

21	Fri, 10/15/2021	Chap. 7	Sturm-Liouville equations -- exam due		
22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021
25	Mon, 10/25/2021	Chap. 5	Rigid body motion	#18	10/29/2021
26	Wed, 10/27/2021	Chap. 8	Elastic two-dimensional membranes		
27	Fri, 10/29/2021	Chap. 9	Mechanics of 3 dimensional fluids		
28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids	#19	11/03/2021
29	Wed, 11/03/2021	Chap. 9	Linearized hydrodynamics equations		

Homework 18 is due Friday.

PHY 711 -- Assignment #19

Nov. 01, 2021

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider the example discussed in class on slides > 11, concerning the flow of an incompressible fluid in the **z** direction in the presence of a stationary cylindrical log oriented in the **y** direction. For this problem, consider the case where the log is replaced by a stationary sphere. Find the velocity potential for this case, using the center of the sphere as the origin of the coordinate system and spherical polar coordinates.

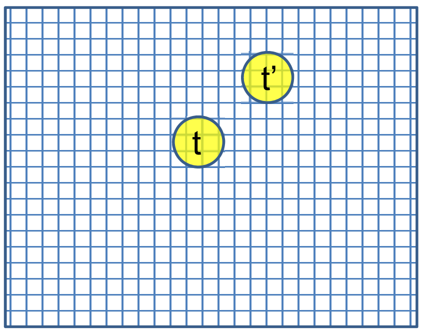
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$



Particle at t: $\mathbf{r}, t$

Particle at t': $\mathbf{r} + \mathbf{v}\delta t, t'$

$$t' = t + \delta t$$

Resuming our discussion of Newton's equations for fluids. For reference, this approach is named for Euler and is based on the continuous fluid being represented within an infinitesimal volume.

Euler analysis -- continued

Particle at t: $\mathbf{r},t$

Particle at t': $\mathbf{r}+\mathbf{v}\delta t, t'$ where $\delta t = t' - t$

For $f(\mathbf{r},t)$:

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{f(\mathbf{r},t') - f(\mathbf{r},t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t,t) - f(\mathbf{r},t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla) f$$

It can be shown that: $(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$

While the infinitesimal volume moves from t to t', the spatial position moves from r to r+v δt

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

For $f \rightarrow v_x \qquad \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + (\mathbf{v} \cdot \nabla) v_x$

For $f \rightarrow v_y \qquad \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + (\mathbf{v} \cdot \nabla) v_y$

For $f \rightarrow v_z \qquad \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + (\mathbf{v} \cdot \nabla) v_z$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}$

Note that $(\mathbf{v} \cdot \nabla)\mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right)\left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right)$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$

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$$= \frac{1}{2}\nabla |\mathbf{v}|^2 - \mathbf{v} \times (\nabla \times \mathbf{v})$$

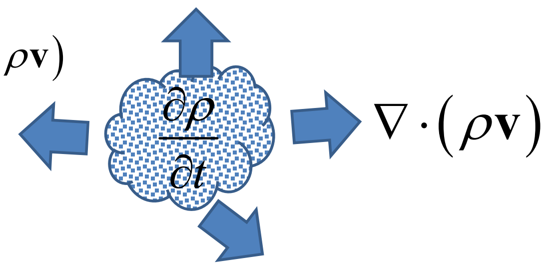
Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

The notion of the continuity is a common feature of continuous closed systems. Here we assume that there are no mechanisms for creation or destruction of the fluid.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

velocity potential

Irrotational flow: $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

For irrotational flow of an incompressible fluid: $\nabla^2 \Phi = 0$

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Another aspect of the fluid is the continuity equation. This simplifies to a velocity field which has zero divergence.
For irrotational flow the velocity field has zero curl and therefore can be written in terms of the velocity potential. Irrotational flow of an incompressible fluid satisfies the Laplace equation.

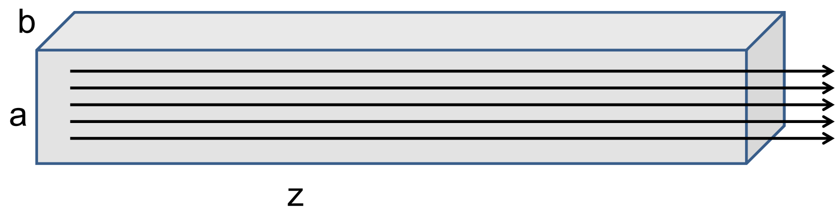
Checking --

Why does $\nabla \times \mathbf{v} = 0$ imply that $\mathbf{v} = -\nabla\Phi$?

Consider: $\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}$

$\nabla \times (\nabla\Phi)\Big|_x = \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial y} = 0$ Similar results for other directions.

Example – uniform flow



$$\nabla^2\Phi = 0$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

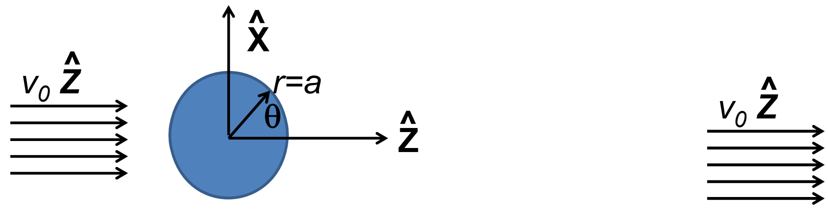
Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

Consider an example of irrotational flow of an incompressible fluid. In this case the fluid is flowing uniformly along the z axis.

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Now imagine the there is a log that distorts the flow. Here the long axis of the log is in the direction perpendicular to the screen. At the boundary of the log, the radial velocity is 0.

Laplace equation in cylindrical coordinates

($r,\theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Setting up and solving the boundary value problem.

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r = a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞: $\Rightarrow A = -v_0$

Some details.

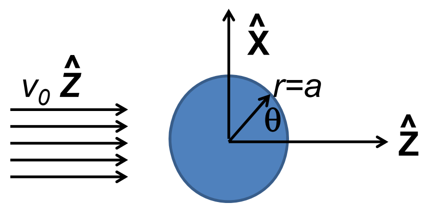
$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

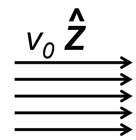
$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For $r \to \infty$

$$\mathbf{v} \to v_0\cos\theta\hat{\mathbf{r}} - v_0\sin\theta\hat{\boldsymbol{\theta}} = v_0\hat{\mathbf{z}}$$





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Full solution and simplified behavior far from the log.

Now consider the case of your homework problem --

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Consider a more complicated situation where there is a pressure gradient and applied potential. Specializing to the case of irrotational flow and arriving at the Bernoulli equation.

For incompressible fluid

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\left(\Phi(\mathbf{r},t)+C(t)\right)$

It is convenient to modify $\Phi(\mathbf{r},t)\to\Phi(\mathbf{r},t)+\int^{t}C(t')dt'$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0$$ Bernoulli's theorem

Not all fluids are compressible, but with additional work we can consider fluids at constant entropy (no heat transfer).

Under what circumstances can there be no heat transfer?

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$dE_{\text{int}} = -dW = -pdV$

Now consider generalizing this result to a possibly compressible fluid under the condition of zero heat transfer (isentropic).

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -p\,dV$$

In terms of mass density: $\rho = \dfrac{M}{V}$

For fixed M and variable V: $d\rho = -\dfrac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -p\,dV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Here we need to introduce the so called first law of thermodynamics. This condition finds a general expression for ratio of the pressure and density in terms of the density derivative of the internal energy density.

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Is this useful?

a. Yes
b. No

This can be rearranged in terms of the gradient of the pressure divided by the density.

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Finally we arrive at a Bernoulli relation for irrotational flow of an isentropic material.

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

Summary of results.