PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 29 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Approximate solutions in the linear limit

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In this lecture, we will continue our discussion of hydrodynamics which is presented in Chapter 9 of your textbook. The focus will be on treating the equations in the linear regime.

22	Mon, 10/18/2021	Chap. 7	Fourier and other transform methods	#15	10/22/2021
23	Wed, 10/20/2021	Chap. 7	Complex variables and contour integration	#16	10/22/2021
24	Fri, 10/22/2021	Chap. 5	Rigid body motion	#17	10/27/2021
25	Mon, 10/25/2021	Chap. 5	Rigid body motion	#18	10/29/2021
26	Wed, 10/27/2021	Chap. 8	Elastic two-dimensional membranes		
27	Fri, 10/29/2021	Chap. 9	Mechanics of 3 dimensional fluids		
28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids	#19	11/03/2021
29	Wed, 11/03/2021	Chap. 9	Linearized hydrodynamics equations	#20	11/05/2021
30	Fri, 11/05/2021	Chap. 9	Linear sound waves		

Updated schedule

PHY 711 -- Assignment #20

Nov. 03, 2021

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Using the analysis covered in class, estimate the speed of sound in the fluid of He gas at 1 atmosphere of pressure and at 300K temperature.

Recall the basic equations of hydrodynamics

Basic variables: Density $\rho(\mathbf{r},t)$

Velocity $\mathbf{v}(\mathbf{r},t)$

Pressure $p(\mathbf{r},t)$

Newton-Euler equation of motion:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

+ relationships among the variables due to principles of thermodynamics due to the particular fluid (In fact, we will focus on an ideal gas.)

Review of the basic equations of hydrodynamics.

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Additional relationships among the variables apply, depending on the fluid material and on thermodynamics

At the moment we are interested in the case where there is no heat exchange.

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$dE_{\text{int}} = -dW = -pdV$ Here W == work
V == volume

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -pdV$$

In terms of mass density: $\rho = \frac{M}{V}$

For fixed M and variable V: $d\rho = -\frac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$ ← Internal energy per unit mass

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Note: Under conditions of constant entropy, we assume e can be expressed in terms of the density alone.

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

if $\nabla\times\mathbf{v} = 0$ ➔ $\mathbf{v} = -\nabla\Phi$ $\mathbf{f}_{applied} = -\nabla U$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic irrotation fluid.

Some details --

$$\left(\nabla \times \mathbf{v}\right) = 0 \qquad \text{"irrotational flow"} \qquad \Rightarrow \mathbf{v} = -\nabla\Phi$$

Check: $\left(\nabla \times \mathbf{v}\right) = -\left(\nabla \times \nabla\Phi\right) = ?$

$$\left(\nabla \times \nabla\Phi\right)\Big|_x = \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial x}$$

Summary: For isentropic and irrotational fluid with internal energy per unit mass ε:

$$\nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

Now consider the fluid to be air near equilibrium

Near equilibrium:

$\rho = \rho_0 + \delta\rho$

$p = p_0 + \delta p$

$\mathbf{v} = 0 + \delta\mathbf{v}$

$\mathbf{f}_{applied} = 0$

ρ_0 represents the average air density

p_0 represents the average air pressure (usually $\approx$ 1 atmosphere)

$\mathbf{v}_0 = 0$ average velocity

Now consider air as the fluid near equilibrium with small fluctuations represented by the delta notation

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}_{applied}-\frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{v}}{\partial t}=-\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0$$

In terms of the velocity potential :

$$\delta\mathbf{v}=-\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t}=-\frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t}+\frac{\delta p}{\rho_0}\right)=0$$

$$\frac{\partial \delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t}-\rho_0\nabla^2\Phi=0$$

Coupled equations.

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho \qquad \text{Here} \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Decoupling the equations in the velocity potential and the density fluctuation.

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla \Phi$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

The decoupled equation is a wave equation in the velocity potential, density fluctuation, and pressure fluctuation variables.

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas:

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38\times 10^{-23} J/K$$

$M_0 =$ average mass of each molecule

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Estimating the wave velocity for air assuming that it is an ideal gas.

Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Using the ideal gas law with f representing the degrees of freedom. It is convenient to replace the f with the gamma factor which can be measured experimentally.

Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma-1} NkT \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0} T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M} dV = \frac{p}{\rho^2} d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Using the ideal gas law under isentropic conditions to derive the speed of sound.

Alternative derivation:

Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

Linearized speed of sound

$$c_0{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s, p_0, \rho_0} = \frac{\gamma p_0}{\rho_0}$$

$$c_0{}^2 \approx \frac{1.5 \cdot 1.013 \times 10^5 \, Pa}{1.3 kg / m^3} \qquad c_0 \approx 340 \text{ m/s}$$

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Another analysis of the speed of sound.

Density dependence of speed of sound for ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0 \gamma}{\rho_0} \frac{p / p_0}{\rho / \rho_0} = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}$$

Some details of the analysis reveal that beyond the linear approximation, the velocity of sound is highly non-linear.

Summary of linearized hydrodynamic equations for isentropic fluid

In terms of the velocity potential:

$$\delta \mathbf{v} = -\nabla \Phi \qquad \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0 \qquad c_0^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s,\rho_0}$$

In term of density fluctuation: $$\frac{\partial^2 \delta\rho}{\partial t^2} - c_0^2 \nabla^2 \delta\rho = 0$$

In term of pressure fluctuation: $$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$